

# FACTORIZATION SYSTEMS INDUCED BY WEAK DISTRIBUTIVE LAWS

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**ABSTRACT.** We relate weak distributive laws in **SetMat** to strictly associative (but not strictly unital) pseudoalgebras of the 2-monad  $(-)^2$  on **Cat**. The corresponding orthogonal factorization systems are characterized by a certain bilinearity property.

## INTRODUCTION

The study of (orthogonal) factorization systems has a long history in category theory, see e.g. [4]. The interest in the subject was renewed by its appearance in Quillen's model categories.

An *orthogonal factorization system* (see e.g. [4]) on a small category  $\mathcal{C}$  is given by subcategories  $\mathcal{E} \hookrightarrow \mathcal{C} \leftarrow \mathcal{M}$ , such that both  $\mathcal{E}$  and  $\mathcal{M}$  contain all isomorphisms of  $\mathcal{C}$ ; any morphism  $f$  in  $\mathcal{C}$  admits a (not necessarily unique) decomposition  $f = me$  in terms of morphisms  $e$  in  $\mathcal{E}$  and  $m$  in  $\mathcal{M}$ ; and the so-called ‘diagonal fill-in’ condition holds, i.e. for any morphisms  $e$  in  $\mathcal{E}$  and  $m$  in  $\mathcal{M}$ , every commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{u} & y \\ e \downarrow & \nearrow t & \downarrow m \\ a & \xrightarrow{v} & b \end{array}$$

in  $\mathcal{C}$  admits a uniquely determined fill-in morphism  $t$  in  $\mathcal{C}$  such that both resulting triangles commute. As it was observed by Korostenski and Tholen in [8], orthogonal factorization systems can be described as pseudoalgebras of the 2-monad  $(-)^2$  on the 2-category **Cat** of categories; functors; and natural transformations.

A *strict factorization system* (see [5]) on  $\mathcal{C}$  means subcategories  $\mathcal{E} \hookrightarrow \mathcal{C} \leftarrow \mathcal{M}$ , such that both  $\mathcal{E}$  and  $\mathcal{M}$  contain all identity morphisms; and any morphism  $f$  in  $\mathcal{C}$  admits a *unique* decomposition  $f = me$  in terms of morphisms  $e$  in  $\mathcal{E}$  and  $m$  in  $\mathcal{M}$ . As it was pointed out by Grandis in [5], strict factorization systems can be regarded as special instances of orthogonal ones – in the sense that any strict factorization system is contained in precisely one orthogonal factorization system. It was proved by Rosebrugh and Wood in [10] that orthogonal factorization systems arising from strict ones are precisely those corresponding to strict algebras for the 2-monad  $(-)^2$  on **Cat** via the correspondence in [8]. Furthermore, in [10] also an equivalence between strict factorization systems and distributive laws (in the sense of [1] or rather of [11]) in the bicategory **SetMat** was established.

Our aim in this paper is to study factorization systems induced by *weak distributive laws* (in the sense introduced by Ross Street in [12]) in **SetMat**. While a distributive

law in **SetMat** determines a strict factorization system; i.e. an inverse of the 2-cell

$$\mathcal{EM} \xrightarrow{\quad} \mathcal{CC} \xrightarrow{\quad \circ \quad} \mathcal{C}$$

in **SetMat**, a weak distributive law corresponds to an appropriate section of this 2-cell, obeying ‘ $\mathcal{E}$ -, and  $\mathcal{M}$ -linearity’ properties (explained in detail in Section 2). We call such a section a *bilinear factorization system*. We prove that there are adjunctions whose counits are isomorphisms, between the following categories. First, the category of weak distributive laws in **SetMat**, to be defined in Section 1; and the category of bilinear factorization systems on small categories, introduced in Section 2. Second, this category of bilinear factorization systems; and the category of strictly associative pseudoalgebras of the 2-monad  $(-)^2$  on **Cat**. Since by results in [8], pseudoalgebras of  $(-)^2$  are equivalent to orthogonal factorization systems, we conclude that any bilinear factorization system is contained in a canonically associated orthogonal factorization system. In particular, weak distributive laws in **SetMat** induce orthogonal factorization systems.

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## 1. WEAK DISTRIBUTIVE LAWS IN **SetMat**

**1.1. Weak distributive laws.** Extending the notion of distributive law due to Jon Beck (see [1]), weak distributive laws in a bicategory were introduced by Ross Street in [12] as follows. They consist of two monads  $(t, \mu, \eta)$ ,  $(s, \nu, \vartheta)$  on the same object, and a 2-cell  $\varphi : ts \rightarrow st$  such that the following diagrams commute (where the coherence 2-cells are omitted, as they are throughout the paper).

$$\begin{array}{ccccc} tts & \xrightarrow{t\varphi} & tst & \xrightarrow{\varphi t} & stt \\ \downarrow \mu s & & \downarrow s\mu & & \\ ts & \xrightarrow{\varphi} & st & & \\ s & \xrightarrow{\eta s} & ts & & \\ \downarrow s\eta & & \downarrow \varphi & & \\ st & \xrightarrow{st\vartheta} & sts & \xrightarrow{s\varphi} & sst & \xrightarrow{\nu t} & st \\ tss & \xrightarrow{\varphi s} & sts & \xrightarrow{s\varphi} & sst \\ \downarrow t\nu & & \downarrow \nu t & & \\ ts & \xrightarrow{\varphi} & st & & \\ t & \xrightarrow{t\vartheta} & ts & & \\ \downarrow \vartheta t & & \downarrow \varphi & & \\ st & \xrightarrow{\eta st} & tst & \xrightarrow{\varphi t} & stt & \xrightarrow{s\mu} & st \end{array}$$

By [12, Proposition 2.2], the second and fourth diagrams can be replaced by a single diagram

$$\begin{array}{ccccc} st & \xrightarrow{\eta st} & tst & \xrightarrow{\varphi t} & stt \\ \downarrow st\vartheta & & & & \downarrow s\mu \\ sts & \xrightarrow{s\varphi} & sst & \xrightarrow{\nu t} & st \end{array} .$$

The equal paths in the latter diagram give rise to an idempotent 2-cell which is an identity if and only if  $\varphi$  is a distributive law in the strict sense.

**1.2. Demimonads.** In simplest terms, a demimonad in a bicategory is a monad in the local Cauchy completion, cf. [9]. Explicitly, it is given by a 1-cell  $t : A \rightarrow A$  and

2-cells  $\mu : t^2 \rightarrow t$  and  $\eta : 1_A \rightarrow t$  such that the following diagrams commute.

$$\begin{array}{ccc}
 \begin{array}{ccc} t^3 & \xrightarrow{\mu t} & t^2 \\ t\mu \downarrow & & \downarrow \mu \\ t^2 & \xrightarrow{\mu} & t \end{array} & 
 \begin{array}{ccc} t & \xrightarrow{\eta t} & t^2 \\ t\eta \downarrow & & \downarrow \mu \\ t^2 & \xrightarrow{\mu} & t \end{array} & 
 \begin{array}{ccc} 1_A & \xrightarrow{\eta\eta} & t^2 \\ & \searrow \eta & \downarrow \mu \\ & & t \end{array} & 
 \begin{array}{ccc} t^2 & \xrightarrow{\eta t^2} & t^3 \\ & \searrow \mu & \downarrow \mu^2 \\ & & t \end{array}
 \end{array}$$

This structure occurred in [2] under the name ‘pre-monad’. The 2-cell  $\mu.\eta t : t \rightarrow t$  is idempotent and whenever it splits, the corresponding retract is a proper monad.

By a special case of [2, Theorem 2.3], for any weak distributive law  $\varphi : ts \rightarrow st$ , the composite 1-cell  $st$  is a demimonad via  $\varphi.\eta\theta : 1_A \rightarrow st$  and  $\nu\mu.s\varphi t : stst \rightarrow st$ .

**1.3. The bicategory  $\mathbf{SetMat}$ .** Recall (e.g. from Section 2 of [3]) that 0-cells in  $\mathbf{SetMat}$  are sets. 1-cells  $f : A \rightarrow B$  are set valued matrices; that is, collections  $\{f(a, b)\}_{a,b}$  of sets labelled by elements  $a$  of  $A$  and  $b$  of  $B$ . 2-cells  $\alpha : f \rightarrow g$  are matrices  $\{\alpha(a, b)\}_{a \in A, b \in B}$  of functions  $\alpha(a, b) : f(a, b) \rightarrow g(a, b)$ . Horizontal composition is given by matrix multiplication; that is,  $hf(a, c) = \sum_{b \in B} h(b, c) \times f(a, b)$ , for 1-cells  $f : A \rightarrow B$  and  $h : B \rightarrow C$ . Horizontal composition of 2-cells is given by the obvious inducement. Vertical composition is given by the elementwise composition of functions.

Functions  $q : A \rightarrow B$  can be regarded as 1-cells  $f_q : A \rightarrow B$  by taking  $f_q(a, b)$  to be a one element set if  $b = q(a)$  and the empty set otherwise.

Small categories are the same as monads in  $\mathbf{SetMat}$ . Functors  $\mathcal{M} \rightarrow \mathcal{M}'$  between them can be described as appropriate monad morphisms (in the sense of [11]) – whose 1-cell part  $f_q$  is induced by the object map  $q$  and whose 2-cell part  $f_q\mathcal{M} \rightarrow \mathcal{M}'f_q$  is given by the morphism map  $\mathcal{M}(a, a') \rightarrow \mathcal{M}'(q(a), q(a'))$ .

**1.4. Weak distributive laws in  $\mathbf{SetMat}$ .** Applying the definition of a weak distributive law in 1.1 to the particular bicategory  $\mathbf{SetMat}$ , we arrive at the following structure. We need two categories  $\mathcal{E}$  and  $\mathcal{M}$  with coinciding object sets  $\mathcal{O}$ , and for all elements  $a, b, c$  of  $\mathcal{O}$  a function

$$\mathcal{E}(b, c) \times \mathcal{M}(a, b) \rightarrow \sum_{x \in \mathcal{O}} \mathcal{M}(x, c) \times \mathcal{E}(a, x), \quad (e, m) \mapsto (e \triangleright m, e \triangleleft m),$$

where  $e \triangleleft m$  denotes a morphism  $a \rightarrow e.m$  in  $\mathcal{E}$ ,  $e \triangleright m$  is a morphism  $e.m \rightarrow c$  in  $\mathcal{M}$  and  $e.m$  is an element of  $\mathcal{O}$ . The four conditions in 1.1 translate to the following requirements, for all morphisms  $e : c \rightarrow b'$  and  $f : b' \rightarrow a'$  in  $\mathcal{E}$  and  $n : a \rightarrow b$  and  $m : b \rightarrow c$  in  $\mathcal{M}$ .

$$\begin{array}{lll}
 (1.1) & 1_c.m = 1_b.1_b & e.1_c = 1_{b'}.1_{b'} \\
 (1.2) & (fe).m = f.(e \triangleright m) & e.(mn) = (e \triangleleft m).n \\
 (1.3) & 1_c \triangleleft m = 1_b \triangleleft 1_b & e \triangleleft 1_c = (1_{b'} \triangleleft 1_{b'})e \\
 (1.4) & (fe) \triangleleft m = [f \triangleleft (e \triangleright m)](e \triangleleft m) & e \triangleleft (mn) = (e \triangleleft m) \triangleleft n \\
 (1.5) & 1_c \triangleright m = m(1_b \triangleright 1_b) & e \triangleright 1_c = 1_{b'} \triangleright 1_{b'} \\
 (1.6) & (fe) \triangleright m = f \triangleright (e \triangleright m) & e \triangleright (mn) = (e \triangleright m)[(e \triangleleft m) \triangleright n].
 \end{array}$$

**1.5. The category induced by a weak distributive law in  $\mathbf{SetMat}$ .** As we recalled in 1.2, for a weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$  in  $\mathbf{SetMat}$ ,  $\mathcal{M}\mathcal{E}$  is a demimonad in  $\mathbf{SetMat}$  via the multiplication  $(n, f)(m, e) = (n(f \triangleright m), (f \triangleleft m)e)$  and

$\eta(a) = (1_a \triangleright 1_a, 1_a \triangleleft 1_a)$ . The corresponding idempotent 2-cell  $\mathcal{M}\mathcal{E} \rightarrow \mathcal{M}\mathcal{E}$  in **SetMat** takes the explicit form

$$(1.7) \quad (m, e) \mapsto (1_c \triangleright m, e \triangleleft 1_a) = (m(1_b \triangleright 1_b), (1_b \triangleleft 1_b)e),$$

for any morphisms  $e : a \rightarrow b$  in  $\mathcal{E}$  and  $m : b \rightarrow c$  in  $\mathcal{M}$ . Since this is clearly split, the corresponding retract is a monad in **SetMat**, that is, a small category. It will be denoted by  $\mathcal{M}_\varrho\mathcal{E}$ . Its objects are the same as the objects of  $\mathcal{E}$  and of  $\mathcal{M}$ , and its morphisms have the form as on the right hand side of (1.7). By the axioms of a demimonad in 1.2, the identity morphism of any object  $a$  is equal to  $\eta(a) = (1_a \triangleright 1_a, 1_a \triangleleft 1_a)$  and the composition law is

$$(1.8) \quad (1_c \triangleright n, f \triangleleft 1_b)(1_b \triangleright m, e \triangleleft 1_a) = (n(f \triangleright m), (f \triangleleft m)e),$$

for morphisms  $e : a \rightarrow x$  and  $f : b \rightarrow y$  in  $\mathcal{E}$  and  $m : x \rightarrow b$  and  $n : y \rightarrow c$  in  $\mathcal{M}$ . (Note that  $1_c \triangleright (n(f \triangleright m)) = n(f \triangleright m)$  and  $((f \triangleleft m)e) \triangleleft 1_a = (f \triangleleft m)e$ .)

Since we are dealing simultaneously with three categories  $\mathcal{E}$ ,  $\mathcal{M}$  and  $\mathcal{M}_\varrho\mathcal{E}$ , the following arrow notation (which we learnt from [10]) will turn out to be handy. For morphisms in  $\mathcal{E}$ , we use arrows of the type  $\twoheadrightarrow$ , for morphisms in  $\mathcal{M}$  we draw  $\triangleright$ . Arrows in  $\mathcal{M}_\varrho\mathcal{E}$  can be drawn as

$$a \xrightarrow{e \triangleleft 1_a} 1_b.1_b \xrightarrow{1_c \triangleright m} c.$$

**Proposition 1.6.** *For a weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$  in **SetMat**, the following functions define identity-on-objects functors  $i : \mathcal{E} \rightarrow \mathcal{M}_\varrho\mathcal{E}$  and  $j : \mathcal{M} \rightarrow \mathcal{M}_\varrho\mathcal{E}$ , respectively.*

$$\begin{aligned} (a \xrightarrow{e} b) &\mapsto (a \xrightarrow{e \triangleleft 1_a} 1_b.1_b \xrightarrow{1_b \triangleright 1_b} b) \quad \text{and} \\ (a \xrightarrow{m} b) &\mapsto (a \xrightarrow{1_a \triangleleft 1_a} 1_a.1_a \xrightarrow{1_b \triangleright m} b). \end{aligned}$$

*Proof.* Preservation of identity morphisms is obvious. In order to verify preservation of compositions, note that  $i(e)$  is the image of  $(1_b, e)$ , and  $j(m)$  is the image of  $(m, 1_a)$ , under the idempotent 2-cell  $\mu.t\eta$  in the demimonad  $t = \mathcal{M}\mathcal{E}$ . Hence preservation of compositions follows from the demimonad identity  $\mu.\mu\mu.t\eta t\eta = \mu$ . When the functor  $\mathcal{E} \rightarrow \mathcal{M}_\varrho\mathcal{E}$  is considered, evaluate this demimonad identity at any object of the form  $(1_c, f, 1_b, e)$  of  $\mathcal{M}\mathcal{E}\mathcal{M}\mathcal{E}(a, c)$ ; and use that by the second equalities in (1.3) and (1.5),  $(1_c, f)(1_b, e) = i(fe)$ . One proceeds symmetrically in case of the functor  $\mathcal{M} \rightarrow \mathcal{M}_\varrho\mathcal{E}$ .  $\square$

**Lemma 1.7.** *For a weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$  in **SetMat**, and for compatible morphisms  $e$  in  $\mathcal{E}$  and  $m$  in  $\mathcal{M}$ , the following diagram in  $\mathcal{M}_\varrho\mathcal{E}$  is commutative,*

$$\begin{array}{ccc} a & \xrightarrow{i(m)} & b \\ i(e \triangleleft m) \downarrow & & \downarrow i(e) \\ e.m & \xrightarrow{i(e \triangleright m)} & c \end{array}$$

where  $i$  denotes both functors in Proposition 1.6.

*Proof.* Apply the demimonad identity  $\mu.\mu\mu.t\eta t\eta = \mu$  to the demimonad  $t = \mathcal{M}\mathcal{E}$  and evaluate it on any element of the form  $(n, 1_b, 1_b, f)$  of  $\mathcal{M}\mathcal{E}\mathcal{M}\mathcal{E}(a, c)$ . It yields

$$(1.9) \quad i(n)i(f) = (a \xrightarrow{f \triangleleft 1_a} 1_b.1_b \xrightarrow{1_c \triangleright n} c).$$

Hence the down-then-right path is equal to the morphism  $(e \triangleright m, e \triangleleft m)$  in  $\mathcal{M}_\varrho\mathcal{E}$ . Evaluating the same demimonad identity now on the element  $(1_c, e, m, 1_a)$  of  $\mathcal{M}\mathcal{E}\mathcal{M}\mathcal{E}(a, c)$ , we conclude that also the right-then-down path is equal to the morphism  $(e \triangleright m, e \triangleleft m)$  in  $\mathcal{M}_\varrho\mathcal{E}$ .  $\square$

**1.8. Injective weak distributive laws in SetMat.** We say that a weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$  in **SetMat** is *injective* provided that both functors  $\mathcal{E} \rightarrow \mathcal{M}_\varrho\mathcal{E}$  and  $\mathcal{M} \rightarrow \mathcal{M}_\varrho\mathcal{E}$  in Proposition 1.6 are inclusions, i.e. they act as injective functions on the morphisms. Equivalently, a weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$  in **SetMat** is injective if and only if  $1_a \triangleleft 1_a$  is a monomorphism in  $\mathcal{E}$  and  $1_a \triangleright 1_a$  is an epimorphism in  $\mathcal{M}$ , for all objects  $a$ .

**1.9. Morphisms between weak distributive laws in SetMat.** A morphism  $(\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}) \rightarrow (\varrho' : \mathcal{E}'\mathcal{M}' \rightarrow \mathcal{M}'\mathcal{E}')$  of weak distributive laws in **SetMat** is by definition a pair of functors  $Q_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{E}'$  and  $Q_\mathcal{M} : \mathcal{M} \rightarrow \mathcal{M}'$  with a common object map  $q$ , such that the following diagram (of 2-cells in **SetMat**) commutes,

$$\begin{array}{ccccc} f_q \mathcal{E}\mathcal{M} & \xrightarrow{Q_\mathcal{E}\mathcal{M}} & \mathcal{E}' f_q \mathcal{M} & \xrightarrow{\mathcal{E}' Q_\mathcal{M}} & \mathcal{E}' \mathcal{M}' f_q \\ f_q \varrho \downarrow & & & & \downarrow \varrho' f_q \\ f_q \mathcal{M}\mathcal{E} & \xrightarrow{Q_\mathcal{M}\mathcal{E}} & \mathcal{M}' f_q \mathcal{E} & \xrightarrow{\mathcal{M}' Q_\mathcal{E}} & \mathcal{M}' \mathcal{E}' f_q \end{array}$$

where  $f_q$  is the 1-cell induced by  $q$ , cf. 1.3. In other words, commutativity of this diagram means the following equalities, for all morphisms  $m : a \rightarrowtail b$  in  $\mathcal{M}$  and  $e : b \twoheadrightarrow c$  in  $\mathcal{E}$ .

$$q(e.m) = Q_\mathcal{E}(e).Q_\mathcal{M}(m), \quad Q_\mathcal{E}(e \triangleleft m) = Q_\mathcal{E}(e) \triangleleft Q_\mathcal{M}(m), \quad Q_\mathcal{M}(e \triangleright m) = Q_\mathcal{E}(e) \triangleright Q_\mathcal{M}(m).$$

Weak distributive laws in **SetMat** and their morphisms constitute a category.

**Proposition 1.10.** *There is an adjunction whose counit is an isomorphism, between the category of all weak distributive laws, and the category of injective weak distributive laws in SetMat.*

*Proof.* A to-be-left-adjoint  $S$  of the inclusion functor  $J$  is constructed as follows. For a weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$ , by Proposition 1.6 there are identity-on-objects functors  $i$  from  $\mathcal{E}$  and from  $\mathcal{M}$  to the associated product category  $\mathcal{M}_\varrho\mathcal{E}$ . Hence we may consider the subcategories  $i(\mathcal{E})$  and  $i(\mathcal{M})$  of  $\mathcal{M}_\varrho\mathcal{E}$ . An (evidently injective) weak distributive law  $S(\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}) := (\tilde{\varrho} : i(\mathcal{E})i(\mathcal{M}) \rightarrow i(\mathcal{M})i(\mathcal{E}))$  is given by

$$\begin{aligned} \tilde{\varrho}(i(e), i(m)) &= \tilde{\varrho}(b \xrightarrow{e \triangleleft 1_b} 1_c.1_c \xrightarrow{1_c \triangleright 1_c} c, a \xrightarrow{1_a \triangleleft 1_a} 1_a.1_a \xrightarrow{1_b \triangleright m} b) := \\ &= (e.m \xrightarrow{1_{e.m} \triangleleft 1_{e.m}} e.m \xrightarrow{e \triangleright m} c, a \xrightarrow{e \triangleleft m} e.m \xrightarrow{1_{e.m} \triangleright 1_{e.m}} e.m) = (i(e \triangleright m), i(e \triangleleft m)). \end{aligned}$$

Observe that the pair  $(i(e \triangleright m), i(e \triangleleft m))$  is uniquely determined by the composite morphism

$$i(e \triangleright m)i(e \triangleleft m) = (e \triangleright m, e \triangleleft m) = i(e)i(m)$$

in  $\mathcal{M}_\varrho\mathcal{E}$ , cf. Lemma 1.7. Thus the 2-cell  $\tilde{\varrho}$  in **SetMat** is well-defined. Since  $i$  stands for functors and  $\varrho$  was a weak distributive law in **SetMat**, so is  $\tilde{\varrho}$  and  $(i : \mathcal{E} \rightarrow i(\mathcal{E}), i : \mathcal{M} \rightarrow i(\mathcal{M}))$  is a morphism of weak distributive laws.

A morphism  $(Q_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{E}', Q_\mathcal{M} : \mathcal{M} \rightarrow \mathcal{M}')$  of weak distributive laws induces functors  $\tilde{Q}_\mathcal{E} : i(\mathcal{E}) \rightarrow i'(\mathcal{E}')$  and  $\tilde{Q}_\mathcal{M} : i(\mathcal{M}) \rightarrow i'(\mathcal{M}')$ ,

$$\begin{aligned}\tilde{Q}_\mathcal{E}(i(e)) &= \tilde{Q}_\mathcal{E}(a \xrightarrow{e \triangleleft 1_a} 1_b \cdot 1_b \xrightarrow{1_b \triangleright 1_b} b) := \\ &= (q(a) \xrightarrow{Q_\mathcal{E}(e \triangleleft 1_a)} q(1_b \cdot 1_b) \xrightarrow{Q_\mathcal{M}(1_b \triangleright 1_b)} q(b)) = i'(Q_\mathcal{E}(e)) \\ \tilde{Q}_\mathcal{M}(i(m)) &= \tilde{Q}_\mathcal{M}(a \xrightarrow{1_a \triangleleft 1_a} 1_a \cdot 1_a \xrightarrow{1_b \triangleright m} b) := \\ &= (q(a) \xrightarrow{Q_\mathcal{E}(1_a \triangleleft 1_a)} q(1_a \cdot 1_a) \xrightarrow{Q_\mathcal{M}(1_b \triangleright m)} q(b)) = i'(Q_\mathcal{M}(m)),\end{aligned}$$

where  $q$  denotes the common object map of the functors  $Q_\mathcal{E}$  and  $Q_\mathcal{M}$ . The last ones of the above equalities are obtained using that  $Q_\mathcal{E}$  and  $Q_\mathcal{M}$  constitute a morphism of weak distributive laws. These functors render commutative the outer square in the following diagram of 2-cells in **SetMat**,

$$\begin{array}{ccccc} f_q \mathcal{E} \mathcal{M} & \xrightarrow{Q_\mathcal{E} \mathcal{M}} & \mathcal{E}' f_q \mathcal{M} & \xrightarrow{\mathcal{E}' Q_\mathcal{M}} & \mathcal{E}' \mathcal{M}' f_q \\ \downarrow f_q \varrho & \searrow f_q i i & \downarrow i' f_q i & \swarrow i' i' f_q & \downarrow \varrho' f_q \\ & f_q i(\mathcal{E}) i(\mathcal{M}) & \xrightarrow{\tilde{Q}_\mathcal{E} i(\mathcal{M})} i'(\mathcal{E}') f_q i(\mathcal{M}) & \xrightarrow{i'(\mathcal{E}') \tilde{Q}_\mathcal{M} i'(\mathcal{M}')} i'(\mathcal{M}') f_q & \\ & \downarrow f_q \tilde{\varrho} & & \downarrow \tilde{\varrho}' f_q & \\ & f_q i(\mathcal{M}) i(\mathcal{E}) & \xrightarrow{\tilde{Q}_\mathcal{M} i(\mathcal{E})} i'(\mathcal{M}') f_q i(\mathcal{E}) & \xrightarrow{i'(\mathcal{M}') \tilde{Q}_\mathcal{E} i'(\mathcal{E}')} i'(\mathcal{M}') i'(\mathcal{E}') f_q & \\ & \swarrow f_q i i & \uparrow i' f_q i & \swarrow i' i' f_q & \\ f_q \mathcal{M} \mathcal{E} & \xrightarrow{Q_\mathcal{M} \mathcal{E}} & \mathcal{M}' f_q \mathcal{E} & \xrightarrow{\mathcal{M}' Q_\mathcal{E}} & \mathcal{M}' \mathcal{E}' f_q \end{array}$$

where  $f_q$  is the 1-cell corresponding to the function  $q$ , see 1.3. Since the 2-cell  $f_q i i$  appearing in the upper left corner is epi, this proves that also the inner square of the diagram commutes; that is,  $S(Q_\mathcal{E}, Q_\mathcal{M}) := (\tilde{Q}_\mathcal{E}, \tilde{Q}_\mathcal{M})$  is a morphism of weak distributive laws. Moreover, there is a commutative square of morphisms of weak distributive laws:

$$\begin{array}{ccc} (\varrho : \mathcal{E} \mathcal{M} \rightarrow \mathcal{M} \mathcal{E}) & \xrightarrow{(i, i)} & (\tilde{\varrho} : i(\mathcal{E}) i(\mathcal{M}) \rightarrow i(\mathcal{M}) i(\mathcal{E})) \\ \downarrow (Q_\mathcal{E}, Q_\mathcal{M}) & & \downarrow (\tilde{Q}_\mathcal{E}, \tilde{Q}_\mathcal{M}) \\ (\varrho' : \mathcal{E}' \mathcal{M}' \rightarrow \mathcal{M}' \mathcal{E}') & \xrightarrow{(i', i')} & (\tilde{\varrho}' : i'(\mathcal{E}') i'(\mathcal{M}') \rightarrow i'(\mathcal{M}') i'(\mathcal{E}')) \end{array}$$

If  $\varrho : \mathcal{E} \mathcal{M} \rightarrow \mathcal{M} \mathcal{E}$  is an injective weak distributive law in **SetMat**, then the functors  $\mathcal{E} \xrightarrow{i} \mathcal{M}_\varrho \mathcal{E} \xleftarrow{i} \mathcal{M}$  in Proposition 1.6 factorize through isomorphisms  $i : \mathcal{E} \rightarrow i(\mathcal{E})$  and  $i : \mathcal{M} \rightarrow i(\mathcal{M})$ . By the above considerations, these amount to an isomorphism  $(\varrho : \mathcal{E} \mathcal{M} \rightarrow \mathcal{M} \mathcal{E}) \rightarrow SJ(\varrho : \mathcal{E} \mathcal{M} \rightarrow \mathcal{M} \mathcal{E})$  of weak distributive laws in **SetMat**, natural with respect to morphisms of weak distributive laws. The counit  $\varepsilon : SJ \rightarrow 1$  of the adjunction is its inverse. The unit  $\eta : 1 \rightarrow JS$  is given by the morphism  $(i : \mathcal{E} \rightarrow i(\mathcal{E}), i : \mathcal{M} \rightarrow i(\mathcal{M}))$  of weak distributive laws, natural in the arbitrary

weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$ . This morphism is taken by  $S$  to the identity morphism  $(i(\mathcal{E}) \rightarrow i(\mathcal{E}), i(\mathcal{M}) \rightarrow i(\mathcal{M}))$  which is equal to the value of  $\varepsilon S$  at the same object. Finally, at an injective weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$ ,  $\eta J$  is the isomorphism  $(\mathcal{E} \rightarrow i(\mathcal{E}), \mathcal{M} \rightarrow i(\mathcal{M}))$ , with the inverse  $J\varepsilon$ .  $\square$

Let us stress that, for an arbitrary weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$ , the injective weak distributive law  $S(\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E})$  constructed in the proof of Proposition 1.10 is still *weak*. This can be seen by noting that the associated idempotent 2-cell  $i(\mathcal{M})i(\mathcal{E}) \rightarrow i(\mathcal{M})i(\mathcal{E})$  in **SetMat** (cf. 1.1) is given by functions

$$(i(m), i(e)) \mapsto (i(1_c \triangleright m), i(e \triangleleft 1_a)),$$

for any morphisms  $e : a \twoheadrightarrow b$  in  $\mathcal{E}$  and  $m : b \rightarrow c$  in  $\mathcal{M}$  – which are not identity functions unless  $b = 1_b \cdot 1_b$ . On the other hand, by the first identity in (1.5), by the second identity in (1.3) and by Lemma 1.7,

$$i(1_c \triangleright m)i(e \triangleleft 1_a) = i(m)i(1_b \triangleright 1_b)i(1_b \triangleleft 1_b)i(e) = i(m)i(1_b)i(1_b)i(e) = i(m)i(e).$$

Thus the product categories  $i(\mathcal{M})_{S(\varrho)}i(\mathcal{E})$  and  $\mathcal{M}_\varrho\mathcal{E}$  coincide.

## 2. BILINEAR FACTORIZATION SYSTEMS

**2.1. Bilinear factorization system.** By a bilinear factorization system on a small category  $\mathcal{C}$  we mean subcategories  $\mathcal{E} \xhookrightarrow{i} \mathcal{C} \xleftarrow{j} \mathcal{M}$ , both containing all identity morphisms, together with a 2-cell  $\delta : \mathcal{C} \rightarrow \mathcal{M}\mathcal{E}$  in **SetMat**,

$$\mathcal{C}(a, b) \rightarrow \sum_x \mathcal{M}(x, b) \times \mathcal{E}(a, x), \quad g \mapsto (\mu(g) : F(g) \rightarrow b, \epsilon(g) : a \twoheadrightarrow F(g)),$$

providing a section for  $\mathcal{M}\mathcal{E} \xrightarrow{ii} \mathcal{C}\mathcal{C} \xrightarrow{\circ} \mathcal{C}$  and rendering commutative the following diagrams.

$$\begin{array}{ccc} \mathcal{M}\mathcal{C} & \xrightarrow{\mathcal{M}\delta} & \mathcal{M}\mathcal{M}\mathcal{E} \\ \downarrow i\mathcal{C} & & \downarrow \circ\mathcal{E} \\ \mathcal{C}\mathcal{C} & & \mathcal{C}\mathcal{C} \\ \downarrow \circ & \xrightarrow{\delta} & \downarrow \delta \\ \mathcal{C} & \xrightarrow{\delta} & \mathcal{M}\mathcal{E} \end{array} \quad \begin{array}{ccc} \mathcal{C}\mathcal{E} & \xrightarrow{\delta\mathcal{E}} & \mathcal{M}\mathcal{E}\mathcal{E} \\ \downarrow \mathcal{C}i & & \downarrow \mathcal{M}\circ \\ \mathcal{C}\mathcal{C} & & \mathcal{C}\mathcal{C} \\ \downarrow \circ & \xrightarrow{\delta} & \downarrow \delta \\ \mathcal{C} & \xrightarrow{\delta} & \mathcal{M}\mathcal{E} \end{array}$$

In other words, for all morphisms  $e : a \twoheadrightarrow b$  in  $\mathcal{E}$ ,  $g : b \rightarrow c$  in  $\mathcal{C}$  and  $m : c \rightarrow d$  in  $\mathcal{M}$ , there is a factorization  $g = i\mu(g)i\epsilon(g)$  and the ‘ $\mathcal{E}$ -, and  $\mathcal{M}$ -linearity conditions’

$$(2.1) \quad \epsilon(i(m)g) = \epsilon(g) \quad \epsilon(gi(e)) = \epsilon(g)e$$

$$(2.2) \quad \mu(i(m)g) = m\mu(g) \quad \mu(gi(e)) = \mu(g)$$

hold.

Clearly, any strict factorization system is bilinear. We could not find in the literature, however, any non-strict factorization system which is bilinear. In order to show that such factorization systems do exist, let stand here some simple examples.

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<sup>1</sup>Throughout, we use the term *subcategory*  $\mathcal{E} \hookrightarrow \mathcal{C}$  ‘up-to isomorphism’ – i.e. in the loose sense that there is an identity-on-objects functor  $i : \mathcal{E} \rightarrow \mathcal{C}$  which acts *injectively* on the morphisms – we do not require  $i$  to be an identity map on the morphisms.



**Example 2.2.** Let  $\mathcal{P}$  be a *proset bounded from below*; that is, a set equipped with a reflexive and transitive relation  $\leq$  and an element  $\perp \in \mathcal{P}$  such that  $\perp \leq a$  for all  $a \in \mathcal{P}$ . For example,  $\mathcal{P}$  can be the set of small sets with  $\leq$  denoting subsets and  $\perp$  being the empty set. As another example,  $\mathcal{P}$  can be chosen to be the set of positive integers with  $p \leq q$  whenever  $p$  divides  $q$ , in which case  $\perp$  stands for the positive integer 1.

Consider an associated category (in fact a groupoid) whose objects are the elements of  $\mathcal{P}$ ; and in which there is precisely one morphism between any pair of objects. Let  $\mathcal{E}$  (resp.  $\mathcal{M}$ ) be the subcategory that contains a morphism  $p \rightarrow q$  if and only if  $q \leq p$  (resp.  $p \leq q$ ). Clearly, both  $\mathcal{E}$  and  $\mathcal{M}$  contain all identity morphisms. But this is not a strict factorization system as there is a morphism  $p \rightarrow l \rightarrow q$  whenever  $l \leq p$  and  $l \leq q$ . This can be made a bilinear factorization system, however, via  $\epsilon(p \rightarrow q) = p \rightarrow \perp$  and  $\mu(p \rightarrow q) = \perp \rightarrow q$ .

**Example 2.3.** Let  $\mathcal{C}$  be a category with two objects 1 and 2, such that there is an isomorphism  $f : 1 \rightarrow 2$ . Let  $\mathcal{E}$  be the subcategory containing all morphisms in  $\mathcal{C}(1, 1) \cup \mathcal{C}(1, 2) \cup \mathcal{C}(2, 2)$  and let  $\mathcal{M}$  be the subcategory containing the identity morphisms  $1_1$  and  $1_2$  and the morphism  $f^{-1} : 2 \rightarrow 1$ . Both  $\mathcal{E}$  and  $\mathcal{M}$  contain all identity morphisms. But this is not a strict factorization system as  $f^{-1}fg = g = 1_1g$  for all  $g \in \mathcal{C}(1, 1)$ . However, it is a bilinear factorization system via

$$\begin{aligned} \epsilon(g) &= \epsilon(fg) = fg & \epsilon(fgf^{-1}) &= \epsilon(gf^{-1}) = fgf^{-1} \\ \mu(g) &= \mu(gf^{-1}) = f^{-1} & \mu(fg) &= \mu(fgf^{-1}) = 1_2, \end{aligned}$$

for all  $g \in \mathcal{C}(1, 1)$ .

**Example 2.4.** Consider a category  $\mathcal{C}$  with three objects 1, 2 and 3 and morphism set generated by three morphisms  $f : 1 \rightarrow 2$ ,  $p : 2 \rightarrow 3$  and  $q : 3 \rightarrow 1$ , modulo the relations  $qp f = 1_1$  and  $fqp = 1_2$ . (That is, the non-identity morphisms in  $\mathcal{C}$  are  $f : 1 \rightarrow 2$ ,  $p : 2 \rightarrow 3$ ,  $q : 3 \rightarrow 1$ ,  $qp = f^{-1} : 2 \rightarrow 1$ ,  $f q : 3 \rightarrow 2$ ,  $p f : 1 \rightarrow 3$  and  $p f q : 3 \rightarrow 3$ .) Let  $\mathcal{E}$  and  $\mathcal{M}$  be the subcategories with the same objects 1, 2 and 3; whose non-identity morphisms are  $\{f, f q\}$  and  $\{p, qp = f^{-1}\}$ , respectively. This is not a strict factorization system since  $f^{-1}f = 1_1 = 1_1 1_1$ . It is, however, a bilinear factorization system via

$$\begin{aligned} \epsilon(1_1) &= \epsilon(f) = \epsilon(p f) = f & \epsilon(q) &= \epsilon(f q) = \epsilon(p f q) = f q \\ \epsilon(1_2) &= \epsilon(p) = \epsilon(q p = f^{-1}) = 1_2 & \epsilon(1_3) &= 1_3 \\ \mu(1_1) &= \mu(q) = \mu(q p = f^{-1}) = f^{-1} & \mu(p) &= \mu(p f) = \mu(p f q) = p \\ \mu(1_2) &= \mu(f) = \mu(f q) = 1_2 & \mu(1_3) &= 1_3. \end{aligned}$$

**2.5. Any weak distributive law in SetMat determines a bilinear factorization system.** Without loss of generality, we may take an injective weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$ . As if for it is not injective, we can replace it with its image under the functor  $S$  in Proposition 1.10. Thus we have subcategories  $\mathcal{E} \xrightarrow{i} \mathcal{M}_\varrho \mathcal{E} \xleftarrow{i} \mathcal{M}$  which contain all identity morphisms. We put

$$\begin{aligned} \epsilon\left(a \xrightarrow{e \triangleleft 1_a} 1_b.1_b \xrightarrow{1_c \triangleright m} c\right) &:= a \xrightarrow{e \triangleleft 1_a} 1_b.1_b \quad \text{and} \\ \mu\left(a \xrightarrow{e \triangleleft 1_a} 1_b.1_b \xrightarrow{1_c \triangleright m} c\right) &:= 1_b.1_b \xrightarrow{1_c \triangleright m} c. \end{aligned}$$



Use (1.9) to see that, for any morphism  $g = (a \xrightarrow{e \triangleleft 1_a} 1_b.1_b \xrightarrow{1_c \triangleright m} c)$  in  $\mathcal{M}_g\mathcal{E}$ ,

$$i\mu(g)i\epsilon(g) = i(1_c \triangleright m)i(e \triangleleft 1_a) = (a \xrightarrow{e \triangleleft 1_a} 1_b.1_b \xrightarrow{1_c \triangleright m} c) = g.$$

Since any morphism in  $\mathcal{M}_g\mathcal{E}$  can be written in the form  $i(m)i(f)$  in terms of some (non-uniquely chosen) morphisms  $f : b \rightarrow c$  in  $\mathcal{E}$  and  $m : c \rightarrow d$  in  $\mathcal{M}$ , the ‘ $\mathcal{E}$ -linearity conditions’ follow for any morphism  $e : a \rightarrow b$  in  $\mathcal{E}$  by

$$\begin{aligned} \epsilon(i(m)i(f)i(e)) &= \epsilon(i(m)i(fe)) = fe \triangleleft 1_a = (f \triangleleft 1_b)e = \epsilon(i(m)i(f))e; \\ \mu(i(m)i(f)i(e)) &= \mu(i(m)i(fe)) = 1_a \triangleright m = \mu(i(m)i(f)). \end{aligned}$$

The third equality in the first line follows by the second condition in (1.3). The ‘ $\mathcal{M}$ -linearity conditions’ are verified symmetrically.

**2.6. Any bilinear factorization system determines a weak distributive law in  $\mathbf{SetMat}$ .** For a bilinear factorization system  $\mathcal{E} \xrightarrow{i} \mathcal{C} \xleftarrow{i} \mathcal{M}$  with  $\delta : \mathcal{C} \rightarrow \mathcal{M}\mathcal{E}$ , we put

$$\varrho := (\mathcal{E}\mathcal{M} \xrightarrow{ii} \mathcal{C}\mathcal{C} \xrightarrow{\circ} \mathcal{C} \xrightarrow{\delta} \mathcal{M}\mathcal{E}).$$

It renders commutative the four diagrams in 1.1:

The left diagram shows the following structure:

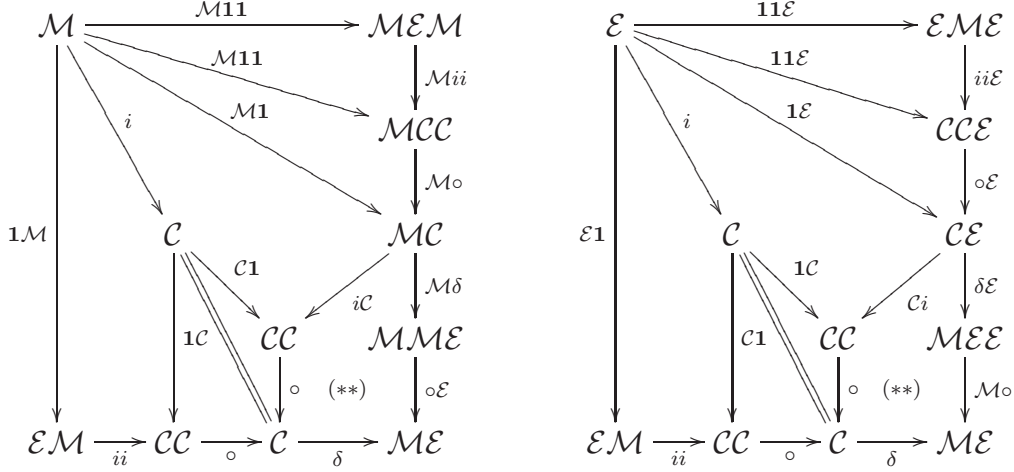
- Top row:  $\mathcal{E}\mathcal{E}\mathcal{M} \xrightarrow{\circ\mathcal{M}} \mathcal{E}\mathcal{M}$
- Second row:  $\mathcal{E}\mathcal{C}\mathcal{C} \xrightarrow{i\mathcal{C}\mathcal{C}} \mathcal{C}\mathcal{C}\mathcal{C} \xrightarrow{\circ\mathcal{C}} \mathcal{C}\mathcal{C}$
- Third row:  $\mathcal{E}\mathcal{C} \xrightarrow{\epsilon\delta} \mathcal{E}\mathcal{M}\mathcal{E} \xrightarrow{\epsilon ii} \mathcal{E}\mathcal{C}\mathcal{C} \xrightarrow{\epsilon\delta} \mathcal{E}\mathcal{C}$
- Fourth row:  $\mathcal{C}\mathcal{C}\mathcal{E} \xrightarrow{\mathcal{C}\mathcal{C}i} \mathcal{C}\mathcal{C}\mathcal{C} \xrightarrow{\mathcal{C}\delta} \mathcal{C}\mathcal{C} \xrightarrow{\circ} \mathcal{C}$
- Fifth row:  $\mathcal{C}\mathcal{E} \xrightarrow{\mathcal{C}i} \mathcal{C}\mathcal{C} \xrightarrow{\delta} \mathcal{M}\mathcal{E}$
- Bottom row:  $\mathcal{M}\mathcal{E}\mathcal{E} \xrightarrow{\mathcal{M}\delta} \mathcal{M}\mathcal{E}$

The right diagram shows the following structure:

- Top row:  $\mathcal{E}\mathcal{M}\mathcal{M} \xrightarrow{\mathcal{E}\delta} \mathcal{E}\mathcal{M}$
- Second row:  $\mathcal{C}\mathcal{C}\mathcal{M} \xrightarrow{\mathcal{C}\mathcal{C}i} \mathcal{C}\mathcal{C}\mathcal{C} \xrightarrow{\mathcal{C}\delta} \mathcal{C}\mathcal{C}$
- Third row:  $\mathcal{C}\mathcal{M} \xrightarrow{\delta\mathcal{M}} \mathcal{M}\mathcal{E}\mathcal{M} \xrightarrow{ii\mathcal{M}} \mathcal{C}\mathcal{C}\mathcal{M} \xrightarrow{\mathcal{C}\delta} \mathcal{C}\mathcal{M}$
- Fourth row:  $\mathcal{M}\mathcal{C}\mathcal{C} \xrightarrow{i\mathcal{C}\mathcal{C}} \mathcal{C}\mathcal{C}\mathcal{C} \xrightarrow{\mathcal{C}\delta} \mathcal{C}\mathcal{C} \xrightarrow{\circ} \mathcal{C}$
- Fifth row:  $\mathcal{M}\mathcal{C} \xrightarrow{i\mathcal{C}} \mathcal{C}\mathcal{C} \xrightarrow{\delta} \mathcal{M}\mathcal{E}$
- Bottom row:  $\mathcal{M}\mathcal{M}\mathcal{E} \xrightarrow{\mathcal{M}\delta} \mathcal{M}\mathcal{E}$

Triangles labeled (\*) commute since  $\delta$  is a section of  $\circ.ii$  and the polygons (\*\*) commute by the  $\mathcal{E}$ -linearity and by the  $\mathcal{M}$ -linearity of  $\delta$ , respectively. The other regions commute since the  $i$ s are functors, since the composition in  $\mathcal{C}$  is associative and by the middle four interchange law in  $\mathbf{SetMat}$ . Moreover, denoting by  $\mathbf{1}$  the unit of a monad in  $\mathbf{SetMat}$  (describing the identity morphisms in the corresponding category),

also the following diagrams commute.



Here again, the same commutative regions  $(**)$  appear. The other regions commute since the  $i$ s are functors, by triviality of compositions with identity morphisms and by the middle four interchange law in **SetMat**.

**Example 2.7.** The bilinear factorization system in Example 2.2 determines a weak distributive law  $\varrho : \mathcal{E}M \rightarrow M\mathcal{E}$  in **SetMat** as follows. For  $a \leq b$  and  $c \leq b$  in  $\mathcal{P}$ , consider  $m : a \rightarrow b$  and  $e : b \rightarrow c$ . Then  $e.m = \perp$ ;  $e \triangleleft m$  is the morphism  $a \rightarrow \perp$  and  $e \triangleright m$  is the morphism  $\perp \rightarrow c$ .

**Example 2.8.** The bilinear factorization system in Example 2.3 determines a weak distributive law  $\varrho : \mathcal{E}M \rightarrow M\mathcal{E}$  in **SetMat** as follows. For all  $g \in \mathcal{C}(1, 1)$ ,

$$\begin{aligned} \varrho(g, 1_1) &= (1 \xrightarrow{fg} 2 \xrightarrow{f^{-1}} 1); & \varrho(g, f^{-1}) &= (2 \xrightarrow{fgf^{-1}} 2 \xrightarrow{f^{-1}} 1); \\ \varrho(fgf^{-1}, 1_2) &= (2 \xrightarrow{fgf^{-1}} 2 \xrightarrow{1_2} 2); & \varrho(fg, 1_1) &= (1 \xrightarrow{fg} 2 \xrightarrow{1_2} 2); \\ \varrho(fg, f^{-1}) &= (2 \xrightarrow{fgf^{-1}} 2 \xrightarrow{1_2} 2). \end{aligned}$$

**Example 2.9.** The bilinear factorization system in Example 2.4 determines a weak distributive law  $\varrho : \mathcal{E}M \rightarrow M\mathcal{E}$  in **SetMat** as follows.

$$\begin{aligned} \varrho(1_1, 1_1) &= (1 \xrightarrow{f} 2 \xrightarrow{f^{-1}} 1); & \varrho(f, 1_1) &= (1 \xrightarrow{f} 2 \xrightarrow{1_2} 2); \\ \varrho(1_2, 1_2) &= (2 \xrightarrow{1_2} 2 \xrightarrow{1_2} 2); & \varrho(p, 1_2) &= (2 \xrightarrow{1_2} 2 \xrightarrow{p} 3); \\ \varrho(f^{-1}, 1_2) &= (2 \xrightarrow{1_2} 2 \xrightarrow{f^{-1}} 1); & \varrho(1_3, 1_3) &= (3 \xrightarrow{1_3} 3 \xrightarrow{1_3} 3); \\ \varrho(q, 1_3) &= (3 \xrightarrow{fq} 2 \xrightarrow{f^{-1}} 1); & \varrho(fq, 1_3) &= (3 \xrightarrow{fq} 2 \xrightarrow{1_2} 2); \\ \varrho(1_3, p) &= (2 \xrightarrow{1_2} 2 \xrightarrow{p} 3); & \varrho(q, p) &= (2 \xrightarrow{1_2} 2 \xrightarrow{f^{-1}} 1); \\ \varrho(fq, p) &= (2 \xrightarrow{1_2} 2 \xrightarrow{1_2} 2); & \varrho(1_1, f^{-1}) &= (2 \xrightarrow{1_2} 2 \xrightarrow{f^{-1}} 1); \\ \varrho(f, f^{-1}) &= (2 \xrightarrow{1_2} 2 \xrightarrow{1_2} 2). \end{aligned}$$

Our next aim is to prove that the constructions in 2.5 and 2.6 are mutual inverses in an appropriate sense.

**Theorem 2.10.** *For any bilinear factorization system  $\mathcal{E} \xrightarrow{i} \mathcal{C} \xleftarrow{i} \mathcal{M}$ ,  $\mathcal{C}$  is isomorphic to the product category  $\mathcal{M}_\rho \mathcal{E}$  corresponding to the weak distributive law  $\rho$  in 2.6.*

*Proof.* The objects of both categories  $\mathcal{C}$  and  $\mathcal{M}_\rho \mathcal{E}$  coincide by construction. As explained in 1.5, the morphisms in  $\mathcal{M}_\rho \mathcal{E}$  are of the form

$$a \xrightarrow{\epsilon i(e)} F(1_b) \xrightarrow{\mu i(m)} c,$$

for morphisms  $e : a \twoheadrightarrow b$  in  $\mathcal{E}$  and  $m : b \rightarrow c$  in  $\mathcal{M}$ , where the same notations are used as in 2.1. The stated isomorphism is given by

$$T : \mathcal{M}_\rho \mathcal{E} \rightarrow \mathcal{C}, \quad (a \xrightarrow{\epsilon i(e)} F(1_b) \xrightarrow{\mu i(m)} c) \mapsto (a \xrightarrow{i \epsilon i(e)} F(1_b) \xrightarrow{i \mu i(m)} c)$$

and its inverse

$$T^{-1} : \mathcal{C} \rightarrow \mathcal{M}_\rho \mathcal{E}, \quad (a \xrightarrow{g} c) \mapsto (a \xrightarrow{\epsilon(g)} F(g) \xrightarrow{\mu(g)} c).$$

$T^{-1}(g)$  is a morphism in  $\mathcal{M}_\rho \mathcal{E}$  as needed, since by the bilinearity properties of  $\epsilon$  and  $\mu$ ,

$$(2.3) \quad \epsilon i \epsilon(g) = \epsilon(i \mu(g) i \epsilon(g)) = \epsilon(g); \quad \mu i \mu(g) = \mu(i \mu(g) i \epsilon(g)) = \mu(g).$$

Evidently,  $T^{-1}$  preserves identity morphisms. It preserves compositions as well, since for any morphisms  $g : a \rightarrow c$  and  $h : c \rightarrow d$  in  $\mathcal{C}$ ,

$$\begin{aligned} T^{-1}(h)T^{-1}(g) &= (\mu(h), \epsilon(h))(\mu(g), \epsilon(g)) = (\mu(h)\mu(i \epsilon(h) i \mu(g)), \epsilon(i \epsilon(h) i \mu(g))\epsilon(g)) \\ &= (\mu(i \mu(h) i \epsilon(h) i \mu(g) i \epsilon(g)), \epsilon(i \mu(h) i \epsilon(h) i \mu(g) i \epsilon(g))) \\ &= (\mu(hg), \epsilon(hg)) = T^{-1}(hg). \end{aligned}$$

The third equality follows by the bilinearity of  $\mu$  and  $\epsilon$ , see (2.1) and (2.2), and the fourth equality follows by the factorization property.

Composites of  $T$  and  $T^{-1}$  in both orders yield identity functors by (2.3) and by the factorization property, respectively:

$$\begin{aligned} T^{-1}T(a \xrightarrow{\epsilon i(e)} F(1_b) \xrightarrow{\mu i(m)} c) &= T^{-1}(a \xrightarrow{i \epsilon i(e)} F(1_b) \xrightarrow{i \mu i(m)} c) = (a \xrightarrow{\epsilon i(e)} F(1_b) \xrightarrow{\mu i(m)} c) \\ TT^{-1}(a \xrightarrow{g} c) &= T(a \xrightarrow{\epsilon(g)} F(g) \xrightarrow{\mu(g)} c) = (a \xrightarrow{g} c). \end{aligned}$$

□

**2.11. Morphisms of bilinear factorization systems.** A morphism  $(\mathcal{E} \xrightarrow{i} \mathcal{C} \xleftarrow{i} \mathcal{M}) \rightarrow (\mathcal{E}' \xrightarrow{i'} \mathcal{C}' \xleftarrow{i'} \mathcal{M}')$  of bilinear factorization systems is by definition a functor  $Q : \mathcal{C} \rightarrow \mathcal{C}'$  which restricts (along the inclusions  $i$ ) to functors  $Q_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{E}'$  and  $Q_\mathcal{M} : \mathcal{M} \rightarrow \mathcal{M}'$  such that

$$Q_\mathcal{E}(\epsilon(g)) = \epsilon'(Q(g)) \quad Q_\mathcal{M}(\mu(g)) = \mu'(Q(g)),$$

for any morphism  $g$  in  $\mathcal{C}$  and  $\mu$  and  $\epsilon$  as in 2.1.

Bilinear factorization systems and their morphisms constitute a category.

**Theorem 2.12.** *The category of injective weak distributive laws in  $\mathbf{SetMat}$  and the category of bilinear factorization systems are equivalent.*

*Proof.* We construct mutually inverse equivalence functors with respective object maps in 2.5 and 2.6.

A functor from the category of bilinear factorization systems to the category of (injective) weak distributive laws is **SetMat** is given by

$$\left( (\mathcal{E} \xrightarrow{i} \mathcal{C} \xleftarrow{i} \mathcal{M}) \xrightarrow{Q} (\mathcal{E}' \xrightarrow{i'} \mathcal{C}' \xleftarrow{i'} \mathcal{M}') \right) \mapsto \left( (\mathcal{E}\mathcal{M} \xrightarrow{\varrho} \mathcal{M}\mathcal{E}) \xrightarrow{(Q_{\mathcal{E}}, Q_{\mathcal{M}})} (\mathcal{E}'\mathcal{M}' \xrightarrow{\varrho'} \mathcal{M}'\mathcal{E}') \right),$$

where the weak distributive laws  $\varrho$  and  $\varrho'$  are constructed as in 2.6 and the functors  $Q_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E}'$  and  $Q_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}'$  are restrictions of  $Q$ . For the functor  $T^{-1}$  in Theorem 2.10,  $T^{-1}i$  is equal to the functor associated in Proposition 1.6 to  $\varrho$ . Hence it follows by Theorem 2.10 that  $\varrho$  is an injective weak distributive law. Since  $Q$  is a morphism of bilinear factorization systems, using the notations in 1.4 and 2.1 we obtain, for any compatible morphisms  $e$  in  $\mathcal{E}$  and  $m$  in  $\mathcal{M}$ ,

$$\begin{aligned} Q_{\mathcal{E}}(e \triangleleft m) &= Q_{\mathcal{E}}(\epsilon(i(e) i(m))) = \epsilon'(Q(i(e) i(m))) = \epsilon'(Q i(e) Q i(m)) \\ &= \epsilon'(i' Q_{\mathcal{E}}(e) i' Q_{\mathcal{M}}(m)) = Q_{\mathcal{E}}(e) \triangleleft Q_{\mathcal{M}}(m) \end{aligned}$$

and symmetrically,  $Q_{\mathcal{M}}(e \triangleright m) = Q_{\mathcal{E}}(e) \triangleright Q_{\mathcal{M}}(m)$ . This proves that  $(Q_{\mathcal{E}}, Q_{\mathcal{M}})$  is a morphism of weak distributive laws.

A functor from the category of injective weak distributive laws in **SetMat** to the category of bilinear factorization systems

$$((\mathcal{E}\mathcal{M} \xrightarrow{\varrho} \mathcal{M}\mathcal{E}) \xrightarrow{(Q_{\mathcal{E}}, Q_{\mathcal{M}})} (\mathcal{E}'\mathcal{M}' \xrightarrow{\varrho'} \mathcal{M}'\mathcal{E}')) \mapsto ((\mathcal{E} \hookrightarrow \mathcal{M}_{\varrho}\mathcal{E} \hookleftarrow \mathcal{M}) \xrightarrow{Q} (\mathcal{E}' \hookrightarrow \mathcal{M}'_{\varrho'}\mathcal{E}' \hookleftarrow \mathcal{M}'))$$

is defined by the functor  $Q : \mathcal{M}_{\varrho}\mathcal{E} \rightarrow \mathcal{M}'_{\varrho'}\mathcal{E}'$ ,

$$\begin{aligned} (2.4) \quad \left( a \xrightarrow{e \triangleleft 1_a} 1_b \cdot 1_b \xrightarrow{1_c \triangleright m} c \right) &\mapsto \left( q(a) \xrightarrow{Q_{\mathcal{E}}(e \triangleleft 1_a)} q(1_b \cdot 1_b) \xrightarrow{Q_{\mathcal{M}}(1_c \triangleright m)} q(c) \right) = \\ &\left( q(a) \xrightarrow{Q_{\mathcal{E}}(e) \triangleleft 1_{q(a)}} 1_{q(b)} \cdot 1_{q(b)} \xrightarrow{1_{q(c)} \triangleright Q_{\mathcal{M}}(m)} q(c) \right), \end{aligned}$$

where  $q$  is the common object map of the functors  $Q_{\mathcal{E}}$  and  $Q_{\mathcal{M}}$  and the notations in 1.4 are used. In order to see that  $Q$  is a morphism of bilinear factorization systems, note that for the functors  $i$  and  $i'$  as in Proposition 1.6 and for any morphism  $e : a \rightarrow b$  in  $\mathcal{E}$ ,

$$\begin{aligned} i' Q_{\mathcal{E}}(e) &= \left( q(a) \xrightarrow{Q_{\mathcal{E}}(e) \triangleleft 1_{q(a)}} 1_{q(b)} \cdot 1_{q(b)} \xrightarrow{1_{q(b)} \triangleright 1_{q(b)}} q(b) \right) \\ &= Q \left( a \xrightarrow{e \triangleleft 1_a} 1_b \cdot 1_b \xrightarrow{1_b \triangleright 1_b} b \right) = Q i(e) \end{aligned}$$

and symmetrically,  $i' Q_{\mathcal{M}}(m) = Q i(m)$  for any morphism  $m$  in  $\mathcal{M}$ . Thus  $Q_{\mathcal{E}}$  and  $Q_{\mathcal{M}}$  are restrictions of  $Q$ . Furthermore, for any morphism  $g = \left( a \xrightarrow{e \triangleleft 1_a} 1_b \cdot 1_b \xrightarrow{1_c \triangleright m} c \right)$  in  $\mathcal{M}_{\varrho}\mathcal{E}$ ,

$$\begin{aligned} Q_{\mathcal{E}}(\epsilon(g)) &= Q_{\mathcal{E}} \left( a \xrightarrow{e \triangleleft 1_a} 1_b \cdot 1_b \right) = \left( q(a) \xrightarrow{Q_{\mathcal{E}}(e) \triangleleft 1_{q(a)}} 1_{q(b)} \cdot 1_{q(b)} \right) \\ &= \epsilon' \left( q(a) \xrightarrow{Q_{\mathcal{E}}(e) \triangleleft 1_{q(a)}} 1_{q(b)} \cdot 1_{q(b)} \xrightarrow{1_{q(c)} \triangleright Q_{\mathcal{M}}(m)} q(c) \right) = \epsilon'(Q(g)) \end{aligned}$$

and symmetrically,  $Q_{\mathcal{M}}(\mu(g)) = \mu'(Q(g))$ .

Iterating both functors above on the category of injective weak distributive laws, we clearly obtain the identity functor. In the opposite order, a morphism  $(\mathcal{E} \xrightarrow{j} \mathcal{C} \xleftarrow{j} \mathcal{M}) \xrightarrow{S} (\mathcal{E}' \xrightarrow{j'} \mathcal{C}' \xleftarrow{j'} \mathcal{M}')$  of bilinear factorization systems is taken to  $(\mathcal{E} \xrightarrow{i} \mathcal{M}_\varrho \mathcal{E} \xleftarrow{i} \mathcal{M}) \xrightarrow{Q} (\mathcal{E}' \xrightarrow{i'} \mathcal{M}'_{\varrho'} \mathcal{E}' \xleftarrow{i'} \mathcal{M}')$ , where the weak distributive laws  $\varrho$  and  $\varrho'$  are of the form in 2.6,  $\mathcal{M}_\varrho \mathcal{E}$  and  $\mathcal{M}'_{\varrho'} \mathcal{E}'$  are the induced product categories and the functors  $i$  and  $i'$  are as in Proposition 1.6. The functor  $Q$  is constructed from the restrictions  $S_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{E}'$  and  $S_\mathcal{M} : \mathcal{M} \rightarrow \mathcal{M}'$  of  $S$  via (2.4). The proof of the theorem is completed by showing that the isomorphism  $T$  in Theorem 2.10 induces a natural isomorphism between this composite functor and the identity functor on the category of bilinear factorization systems. The functor  $T : \mathcal{M}_\varrho \mathcal{E} \rightarrow \mathcal{C}$  (and hence also its inverse) restricts to identity functors on  $\mathcal{E}$  and  $\mathcal{M}$ . That is, for any morphism  $e : a \rightarrow b$  in  $\mathcal{E}$ ,

$$\begin{aligned} Ti(e) &= T(a \xrightarrow{\epsilon j(e)} F(1_b) \xrightarrow{\mu(1_b)} b) = (a \xrightarrow{j\epsilon j(e)} F(1_b) \xrightarrow{j\mu(1_b)} b) \\ &= (a \xrightarrow{j(e)} b \xrightarrow{j\epsilon(1_b)} F(1_b) \xrightarrow{j\mu(1_b)} b) = j(e), \end{aligned}$$

where the penultimate equality follows by the second identity in (2.1). Symmetrically,  $Ti(m) = j(m)$  for any morphism  $m$  in  $\mathcal{M}$ . Moreover, for any morphism  $g = (a \xrightarrow{\epsilon j(e)} F(1_b) \xrightarrow{\mu j(m)} c)$  in  $\mathcal{M}_\varrho \mathcal{E}$ ,

$$\epsilon(T(g)) = \epsilon(a \xrightarrow{j\epsilon j(e)} F(1_b) \xrightarrow{j\mu j(m)} c) = (a \xrightarrow{\epsilon j(e)} F(1_b)) = \epsilon_{\mathcal{M}_\varrho \mathcal{E}}(g),$$

where the penultimate equality follows by the bilinearity properties of  $\epsilon$ ; cf. (2.3). Symmetrically, also  $\mu(T(g)) = \mu_{\mathcal{M}_\varrho \mathcal{E}}(g)$ . This proves that  $T$  is an isomorphism of bilinear factorization systems.

Its naturality, i.e. the equality of functors  $ST = T'Q$  is immediate.  $\square$

As a consequence of Proposition 1.10 and Theorem 2.12, we obtain the following.

**Corollary 2.13.** *There is an adjunction whose counit is an isomorphism, between the category of weak distributive laws in **SetMat** and the category of bilinear factorization systems.*

### 3. STRICTLY ASSOCIATIVE PSEUDOALGEBRAS FOR THE 2-MONAD $(-)^2$ ON **Cat**

**3.1. The 2-monad  $(-)^2$  on **Cat**.** Recall (e.g. from Section 1 of [8]) that the 2-functor  $(-)^2$  on the 2-category **Cat** of categories; functors; and natural transformations, is given as follows. For any category  $\mathcal{C}$ , the objects of  $\mathcal{C}^2$  are morphisms in  $\mathcal{C}$ . Morphisms from  $f : a \rightarrow x$  to  $g : b \rightarrow y$  are pairs of morphisms  $u : a \rightarrow b$  and  $v : x \rightarrow y$  such that the first diagram in

(3.1)

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ f \downarrow & & \downarrow g \\ x & \xrightarrow{v} & y \end{array} \quad F(u, v) = \begin{array}{ccc} F(a) & \xrightarrow{F(u)} & F(b) \\ F(f) \downarrow & & \downarrow F(g) \\ F(x) & \xrightarrow{F(v)} & F(y) \end{array} \quad \begin{array}{ccc} F(a) & \xrightarrow{\omega_a} & G(a) \\ F(f) \downarrow & & \downarrow G(f) \\ F(x) & \xrightarrow{\omega_x} & G(x) \end{array}$$

commutes. For a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $F^2 : \mathcal{C}^2 \rightarrow \mathcal{D}^2$  takes the morphism in the first figure to the morphism in  $\mathcal{D}^2$  in the second figure of (3.1). For a natural transformation  $\omega : F \rightarrow G$  and an object  $f : a \rightarrow x$  of  $\mathcal{C}^2$  (i.e. a morphism  $f$  in  $\mathcal{C}$ ),  $\omega_f^2$  is the morphism in  $\mathcal{D}^2$  depicted in the third figure of (3.1).

Multiplication of the 2-monad  $(-)^2$  is given at any category  $\mathcal{C}$  by the functor  $M_{\mathcal{C}} : \mathcal{C}^{2 \times 2} \rightarrow \mathcal{C}^2$  with the morphism map

$$(3.2) \quad \begin{array}{ccccc} & & a' & \xrightarrow{u'} & b' \\ & \nearrow k & \downarrow f' & \searrow l & \downarrow g' \\ a & \xrightarrow{u} & b & \xrightarrow{m} & x' & \xrightarrow{v'} & y' \\ & \searrow v & \downarrow g & \nearrow n & & & \\ x & & y & & & & \end{array} \mapsto \begin{array}{ccc} a & \xrightarrow{k} & a' \\ \downarrow v f = g u & & \downarrow v' f' = g' u' \\ y & \xrightarrow{n} & y' \end{array}$$

and the unit is given by the functor  $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}^2$  with the morphism map

$$a \xrightarrow{f} x \quad \mapsto \quad \begin{array}{ccc} a & \xrightarrow{f} & x \\ 1_a \downarrow & & \downarrow 1_x \\ a & \xrightarrow{f} & x \end{array}.$$

**3.2. (Strictly associative) pseudoalgebras of the 2-monad  $(-)^2$ .** Applying the definition of a pseudoalgebra (see e.g. [6]) to the particular 2-monad  $(-)^2$ , the following notion is obtained. It is given by a category  $\mathcal{C}$ , a functor  $F : \mathcal{C}^2 \rightarrow \mathcal{C}$  and natural isomorphisms  $\vartheta : F I_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$  and  $\varphi : F M_{\mathcal{C}} \rightarrow F F^2$  such that the following diagrams (of natural transformations) commute.

$$\begin{array}{ccc} F \xrightarrow{\varphi I_{\mathcal{C}}^2} F F^2 I_{\mathcal{C}}^2 & & F M_{\mathcal{C}} M_{\mathcal{C}}^2 \xrightarrow{\varphi M_{\mathcal{C}}^2} F F^2 M_{\mathcal{C}}^2 = F(F M_{\mathcal{C}})^2 \\ \parallel & \searrow & \parallel \\ F M_{\mathcal{C}} I_{\mathcal{C}}^2 & & F M_{\mathcal{C}} M_{\mathcal{C}}^2 \\ \downarrow \varphi I_{\mathcal{C}^2} & & \downarrow \varphi M_{\mathcal{C}^2} \\ F F^2 I_{\mathcal{C}}^2 = F I_{\mathcal{C}} F & \xrightarrow{\vartheta F} & F \\ & & \downarrow F \vartheta^2 \\ & & F(F I_{\mathcal{C}})^2 \end{array} \quad \begin{array}{ccc} F F^2 M_{\mathcal{C}}^2 = F M_{\mathcal{C}} F^{2 \times 2} & \xrightarrow{\varphi F^{2 \times 2}} & F F^2 F^{2 \times 2} \\ & & \parallel \\ & & F(F F^2)^2 \\ & & \downarrow F \varphi^2 \end{array}$$

We say that a pseudoalgebra  $(\mathcal{C}, F, \vartheta, \varphi)$  is strictly associative whenever  $\varphi$  is the identity – hence the natural isomorphism  $\vartheta : F I_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$  satisfies

$$(3.3) \quad F \vartheta^2 = 1_F = \vartheta F.$$

Strict associativity of  $F$  explicitly means the equality

$$(3.4) \quad \begin{array}{ccc} F(f) & \xrightarrow{F(k,m)} & F(f') \\ F(u,v) \downarrow & & \downarrow F(u',v') \\ F(g) & \xrightarrow{F(l,n)} & F(g') \end{array} = F \left( \begin{array}{ccc} a & \xrightarrow{k} & a' \\ g u \downarrow & & \downarrow g' u' \\ y & \xrightarrow{n} & y' \end{array} \right)$$

of morphisms in  $\mathcal{C}$ , for any morphism in  $\mathcal{C}^{2 \times 2}$  as on the left hand side of (3.2).

As explained in Section 2.2 of [8], any pseudoalgebra  $(\mathcal{C}, F, \vartheta, \varphi)$  of  $(-)^2$  can be modified such that it becomes strictly unital, i.e. such that the natural isomorphism

$\vartheta : FI_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$  becomes the identity. This modification certainly affects the associativity natural isomorphism  $\varphi : FM_{\mathcal{C}} \rightarrow FF^2$  as well. Our functorial constructions later in this section, however, lead to strictly associative but not necessarily strictly unital pseudoalgebras. We prefer not to modify them to be strictly unital (spoiling perhaps their strict associativity). As we will see, in our approach the unitality natural isomorphism  $\vartheta$  arises in a natural way.

**3.3. Any bilinear factorization system determines a strictly associative pseudoalgebra of the 2-monad  $(-)^2$ .** This construction extends that in [10]. If  $\mathcal{E} \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{M}$  is a bilinear factorization system, then applying the notations in 2.1, for any morphism in  $\mathcal{C}^2$  as in the first diagram in (3.1), we can draw a fully commutative diagram in  $\mathcal{C}$ :

$$\begin{array}{ccccc}
 a & \xrightarrow{i\epsilon(u)} & F(u) & \xrightarrow{i\mu(u)} & b \\
 i\epsilon(f) \downarrow & & \downarrow i\epsilon(i\epsilon(g) i\mu(u)) & & \downarrow i\epsilon(g) \\
 F(f) & \xrightarrow{i\epsilon(i\epsilon(v) i\mu(f))} & F(vf) = F(gu) & \xrightarrow{i\mu(i\epsilon(g) i\mu(u))} & F(g) \\
 i\mu(f) \downarrow & & \downarrow i\mu(i\epsilon(v) i\mu(f)) & & \downarrow i\mu(g) \\
 x & \xrightarrow{i\epsilon(v)} & F(v) & \xrightarrow{i\mu(v)} & y
 \end{array}$$

Indeed, the upper right square commutes since the path down-then-right provides a factorization of the path right-then-down. By the same reasoning, also the bottom left square is commutative. Combining the factorization property with the first equality in (2.1) and with the second equality in (2.2), respectively, we conclude that for any morphisms  $f : a \rightarrow b$  and  $g : b \rightarrow c$  in  $\mathcal{C}$ ,

$$(3.5) \quad \epsilon(gf) = \epsilon(i\mu(g) i\epsilon(g) f) = \epsilon(i\epsilon(g) f); \quad \mu(gf) = \mu(g i\mu(f) i\epsilon(f)) = \mu(g i\mu(f)).$$

Applying this together with the second equality in (2.1), we see that the right-then-down path in the top left square is equal to

$$i\epsilon(i\epsilon(g) i\mu(u)) i\epsilon(u) = i\epsilon(g i\mu(u) i\epsilon(u)) = i\epsilon(gu)$$

while the down-then-right path is equal to

$$i\epsilon(i\epsilon(v) i\mu(f)) i\epsilon(f) = i\epsilon(v i\mu(f) i\epsilon(f)) = i\epsilon(vf).$$

Hence also the top left square commutes and symmetrically so does the bottom right square.

With this information at hand, we define a functor  $F : \mathcal{C}^2 \rightarrow \mathcal{C}$  with the morphism map

$$\begin{array}{ccc}
 a & \xrightarrow{u} & b \\
 f \downarrow & & \downarrow g \\
 x & \xrightarrow{v} & y
 \end{array} \mapsto F(f) \xrightarrow{i\epsilon(i\epsilon(v) i\mu(f))} F(vf) = F(gu) \xrightarrow{i\mu(i\epsilon(g) i\mu(u))} F(g).$$

In order to see that  $F$  preserves identity morphisms, use (3.5); (2.1) and (2.2); and the factorization property to deduce, for any morphism  $f : a \rightarrow x$  in  $\mathcal{C}$ ,

$$F(1_f) = i\mu(i\epsilon(f) i\mu(1_a)) i\epsilon(i\epsilon(1_x) i\mu(f)) = i\mu i\epsilon(f) i\epsilon i\mu(f) = i\mu(1_{F(f)}) i\epsilon(1_{F(f)}) = 1_{F(f)}.$$



For any commutative diagram

$$\begin{array}{ccccc} & & u & & w \\ & \nearrow & & \searrow & \\ f \downarrow & & & & \downarrow h \\ & \searrow & & \nearrow & \\ & & v & & t \end{array}$$

in  $\mathcal{C}$ , it follows by (3.5) that

$$F(wu, tv) = i\mu(i\epsilon(h)wu) i\epsilon(tv i\mu(f)).$$

On the other hand, by applying (3.5) (in the first equality) and the factorization property (in the second equality), we obtain

$$\begin{aligned} F(w, t)F(u, v) &= i\mu(i\epsilon(h) i\mu(w)) i\epsilon(t i\mu(g)) i\mu(i\epsilon(g)u) i\epsilon(i\epsilon(v) i\mu(f)) \\ &= i\mu(i\epsilon(h) i\mu(w)) i\mu(i\epsilon(t i\mu(g)) i\mu(i\epsilon(g)u)) \cdot \\ &\quad i\epsilon(i\epsilon(t i\mu(g)) i\mu(i\epsilon(g)u)) i\epsilon(i\epsilon(v) i\mu(f)). \end{aligned}$$

In order to compare these expressions, note that by (3.5); the second condition in (2.1) and the factorization property; and since  $tg = hw$ ,

$$i\mu(i\epsilon(t i\mu(g)) i\mu(i\epsilon(g)u)) = i\mu(i\epsilon(t i\mu(g)) i\epsilon(g)u) = i\mu(i\epsilon(tg)u) = i\mu(i\epsilon(hw)u).$$

By the factorization property and the first condition in (2.2); by (3.5); and by the second condition in (2.1) and the factorization property,

$$\begin{aligned} i\mu(i\epsilon(h)wu) &= i\mu(i\epsilon(h) i\mu(w)) i\mu(i\epsilon(i\epsilon(h) i\mu(w)) i\epsilon(w)u) \\ &= i\mu(i\epsilon(h) i\mu(w)) i\mu(i\epsilon(h i\mu(w)) i\epsilon(w)u) \\ &= i\mu(i\epsilon(h) i\mu(w)) i\mu(i\epsilon(hw)u). \end{aligned}$$

This shows that  $i\mu(i\epsilon(h) i\mu(w)) i\mu(i\epsilon(t i\mu(g)) i\mu(i\epsilon(g)u)) = i\mu(i\epsilon(h)wu)$ . By symmetrical considerations,  $i\epsilon(i\epsilon(t i\mu(g)) i\mu(i\epsilon(g)u)) i\epsilon(i\epsilon(v) i\mu(f)) = i\epsilon(tv i\mu(f))$  – proving that  $F$  preserves composition as well.

In order to see strict associativity of  $F$ , consider the morphism in  $\mathcal{C}^{2 \times 2}$  on the left hand side of (3.2). The functor  $FM_{\mathcal{C}}$  takes it to

$$i\mu(i\epsilon(g'u') i\mu(k)) i\epsilon(i\epsilon(n) i\mu(gu))$$

while  $FF^2$  takes it to

$$i\mu(i\epsilon(i\epsilon(v') i\mu(f')) i\mu(i\epsilon(f') i\mu(k))) i\epsilon(i\epsilon(i\epsilon(n) i\mu(g)) i\mu(i\epsilon(g) i\mu(u))).$$

These are equal morphisms  $F(gu) \rightarrow F(g'u')$  in  $\mathcal{C}$  since by (3.5); (2.1) and the factorization property,

$$\begin{aligned} i\mu(i\epsilon(i\epsilon(v') i\mu(f')) i\mu(i\epsilon(f') i\mu(k))) &= i\mu(i\epsilon(v' i\mu(f')) i\epsilon(f')k) \\ &= i\mu(i\epsilon(v' f')k) = i\mu(i\epsilon(g'u')k), \end{aligned}$$

and symmetrically,  $i\epsilon(i\epsilon(i\epsilon(n) i\mu(g)) i\mu(i\epsilon(g) i\mu(u))) = i\epsilon(i\epsilon(n) i\mu(gu))$ .

The coherence natural isomorphism  $\vartheta : FI_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$  is given by the isomorphism  $\vartheta_x = i\mu(1_x)$ , with the inverse  $\vartheta_x^{-1} = i\epsilon(1_x)$ , for any object  $x$  of  $\mathcal{C}$ . Indeed,  $\vartheta_x \vartheta_x^{-1} =$

$i\mu(1_x)i\epsilon(1_x) = 1_x$  by the factorization property; and combining the factorization property with (3.5), (2.1) and (2.2), also

$$\begin{aligned}\vartheta_x^{-1}\vartheta_x &= i\mu(i\epsilon(1_x)i\mu(1_x))i\epsilon(i\epsilon(1_x)i\mu(1_x)) \\ &= i\mu i\epsilon(1_x)i\epsilon i\mu(1_x) = i\mu(1_{F(1_x)})i\epsilon(1_{F(1_x)}) = 1_{F(1_x)}.\end{aligned}$$

Naturality of  $\vartheta$  follows by the equality of

$$\begin{aligned}\vartheta_x FI_{\mathcal{C}}(f) &= i\mu(1_x)i\mu(i\epsilon(1_x)i\mu(f))i\epsilon(i\epsilon(f)i\mu(1_a)) = i\mu(f)i\epsilon(fi\mu(1_a)) \quad \text{and} \\ f\vartheta_a &= fi\mu(1_a) = i\mu(fi\mu(1_a))i\epsilon(fi\mu(1_a)) = i\mu(f)i\epsilon(fi\mu(1_a)),\end{aligned}$$

for any morphism  $f : a \rightarrow x$  in  $\mathcal{C}$ . Thus  $FI_{\mathcal{C}}(f) = i\epsilon(1_x)fi\mu(1_a)$ . Finally,  $\vartheta$  obeys the coherence conditions (3.3) since by the bilinearity conditions (2.1) and (2.2),  $F(f) = F(i\mu(f)1_{F(f)}i\epsilon(f)) = F(1_{F(f)})$  and thus

$$\begin{aligned}\vartheta_{F(f)} &= i\mu(1_{F(f)}) = i\mu i\mu(1_{F(f)})i\epsilon i\mu(1_{F(f)}) = i\mu(1_{F(f)})i\epsilon(1_{F(f)}) = 1_{F(f)}; \\ F(\vartheta_f^2) &= i\mu(i\epsilon(f)i\mu i\mu(1_a))i\epsilon(i\epsilon i\mu(1_x)i\mu(i\epsilon(1_x)fi\mu(1_a))) = i\mu i\epsilon(f)i\epsilon i\mu(f) \\ &= i\mu(1_{F(f)})i\epsilon(1_{F(f)}) = 1_{F(f)}.\end{aligned}$$

**3.4. Any strictly associative pseudoalgebra of the 2-monad  $(-)^2$  determines a bilinear factorization system.** This construction, again, extends that in [10]. Consider a strictly associative pseudoalgebra  $(\mathcal{C}, F)$  of the 2-monad  $(-)^2$ , with coherence natural isomorphism  $\vartheta : FI_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$ . For any morphism  $f : a \rightarrow x$  in  $\mathcal{C}$ , the morphism  $I_{\mathcal{C}}(f)$  in  $\mathcal{C}^2$  has a factorization

$$I_{\mathcal{C}}(f) = \begin{array}{ccc} a & \xrightarrow{f} & x \\ 1_a \downarrow & & \downarrow 1_x \\ a & \xrightarrow{f} & x \end{array} = \begin{array}{ccccc} a & \xrightarrow{1_a} & a & \xrightarrow{f} & x \\ 1_a \downarrow & & \downarrow f & & \downarrow 1_x \\ a & \xrightarrow{f} & x & \xrightarrow{1_x} & x \end{array}$$

Applying  $F$  to this equality, we can write  $f$  in the equal form

$$(3.6) \quad f = \vartheta_x FI_{\mathcal{C}}(f)\vartheta_a^{-1} = (a \xrightarrow{\vartheta_a^{-1}} FI_{\mathcal{C}}(a) \xrightarrow{F(1_a, f)} F(f) \xrightarrow{F(f, 1_x)} FI_{\mathcal{C}}(x) \xrightarrow{\vartheta_x} x).$$

Using this decomposition of  $f$ , we construct two subcategories  $\mathcal{E}$  and  $\mathcal{M}$  of  $\mathcal{C}$  as follows. Objects in both subcategories coincide with the objects in  $\mathcal{C}$ . Morphisms  $a \rightarrow x$  in  $\mathcal{E}$  are those morphisms  $e$  in  $\mathcal{C}$  for which  $F(e) = FI_{\mathcal{C}}(x)$  and  $F(e, 1_x) = 1_{FI_{\mathcal{C}}(x)}$ . That is, morphisms of the form

$$(3.7) \quad e = (a \xrightarrow{\vartheta_a^{-1}} FI_{\mathcal{C}}(a) \xrightarrow{F(1_a, f)} F(f) = FI_{\mathcal{C}}(x) \xrightarrow{\vartheta_x} x),$$

where  $f : a \rightarrow x$  is any morphism in  $\mathcal{C}$  such that  $F(f) = FI_{\mathcal{C}}(x)$ . Indeed, for  $e$  as in

$$(3.7), \quad FI_{\mathcal{C}}(e) = (F(1_a) \xrightarrow{F(1_a, f)} F(f)). \quad \text{Hence } F(e, 1_x) \text{ is equal to}$$

$$F\left(\begin{array}{ccccc} a & \xrightarrow{\vartheta_a^{-1}} & F(1_a) & \xrightarrow{F(1_a, f)} & F(1_x) & \xrightarrow{\vartheta_x} & x \\ \downarrow e & & \downarrow F(1_a, f) & & \downarrow 1_{F(1_x)} & & \downarrow 1_x \\ x & \xrightarrow{\vartheta_x^{-1}} & F(f) & \xrightarrow{1_{F(1_x)}} & F(1_x) & \xrightarrow{\vartheta_x} & x \end{array}\right) = F\left(\begin{array}{ccc} F(1_a) & \xrightarrow{F(1_a, f)} & F(f) \\ \downarrow F(1_a, f) & & \downarrow F(1_a, 1_x) \\ F(f) & \xrightarrow{F(1_a, 1_x)} & F(f) \end{array}\right) = F\left(\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ \downarrow f & & \downarrow f \\ x & \xrightarrow{1_x} & x \end{array}\right),$$

i.e. to  $1_{F(f)} = 1_{FI_{\mathcal{C}}(x)}$ . The first equality follows by the coherence condition  $F\vartheta^2 = 1_F$  and the second one follows by the associativity of  $F$  cf. (3.4). Symmetrically, the

morphisms  $a \rightarrow x$  in  $\mathcal{M}$  are those morphisms  $m$  in  $\mathcal{C}$  for which  $F(m) = FI_{\mathcal{C}}(a)$  and  $F(1_a, m) = 1_{FI_{\mathcal{C}}(a)}$ . That is, they are of the form

$$a \xrightarrow{\vartheta_a^{-1}} FI_{\mathcal{C}}(a) = F(f) \xrightarrow{F(f, 1_x)} FI_{\mathcal{C}}(x) \xrightarrow{\vartheta_x} x,$$

for morphisms  $f : a \rightarrow x$  in  $\mathcal{C}$  such that  $F(f) = FI_{\mathcal{C}}(a)$ . The identity morphisms are clearly contained both in  $\mathcal{E}$  and  $\mathcal{M}$ ; we need to check that they are closed under composition. For that the following lemma will be needed.

**Lemma.** *For any morphisms  $f : y \rightarrow x$  in  $\mathcal{C}$  and  $m : x \rightarrow z$  in  $\mathcal{M}$ , the following hold.*

- (a)  $F(mf) = F(f)$ ;
- (b)  $(F(f) \xrightarrow{F(1_y, m)} F(mf)) = 1_{F(f)}$ ;
- (c)  $(F(mf) \xrightarrow{F(f, 1_z)} F(m)) = (F(f) \xrightarrow{F(f, 1_x)} F(1_x))$ .

*Symmetrically, for any morphisms  $e : x \rightarrow y$  in  $\mathcal{E}$  and  $f : y \rightarrow z$  in  $\mathcal{C}$ , the following hold.*

- (a')  $F(fe) = F(f)$ ;
- (b')  $(F(fe) \xrightarrow{F(e, 1_z)} F(f)) = 1_{F(f)}$ ;
- (c')  $(F(e) \xrightarrow{F(1_x, f)} F(fe)) = (F(1_y) \xrightarrow{F(1_y, f)} F(f))$ .

*Proof.* Evaluating the associativity constraint (3.4) on the morphism in  $\mathcal{C}^{2 \times 2}$  depicted in the first figure below, we obtain the equality in the second figure.

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & y & \xrightarrow{f} & x \\
 & \nearrow 1_y & \downarrow 1_y & \nearrow 1_x & \downarrow m \\
 y & \xrightarrow{f} & x & \xrightarrow{mf} & z \\
 \downarrow 1_y & & \downarrow 1_x & & \downarrow m \\
 y & \xrightarrow{f} & x & \xrightarrow{m} & z
 \end{array}
 & = &
 \begin{array}{ccc}
 F(1_y) & \xrightarrow{F(1_y, 1_y)} & F(1_y) \\
 \downarrow F(f, f) & & \downarrow F(f, mf) \\
 F(1_x) & \xrightarrow{F(1_x, m)} & F(m)
 \end{array}
 = F \left( \begin{array}{ccc}
 y & \xrightarrow{1_y} & y \\
 \downarrow f & & \downarrow mf \\
 x & \xrightarrow{m} & z
 \end{array} \right)
 \end{array}$$

Since  $m$  is a morphism in  $\mathcal{M}$ , on the left hand side of this equality the bottom arrow is the identity arrow  $F(1_x, 1_x)$  and both vertical arrows are equal. Hence the left hand side is equal to  $F(1_{F(f, f)}) = 1_{F(f)}$ , proving assertions (a) and (b).

Applying  $F^2$  to the morphism in  $\mathcal{C}^{2 \times 2}$  in the first figure below, we obtain the morphism in  $\mathcal{C}^2$  in the second figure.

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & y & \xrightarrow{f} & x \\
 & \nearrow 1_y & \downarrow mf & \nearrow 1_x & \downarrow m \\
 y & \xrightarrow{f} & x & \xrightarrow{mf} & z \\
 \downarrow f & & \downarrow 1_x & & \downarrow m \\
 x & \xrightarrow{1_x} & x & \xrightarrow{m} & z
 \end{array}
 & = &
 \begin{array}{ccc}
 F(f) & \xrightarrow{F(1_y, m)} & F(mf) \\
 \downarrow F(f, 1_x) & & \downarrow F(f, 1_z) \\
 F(1_x) & \xrightarrow{F(1_x, m)} & F(m)
 \end{array}
 \end{array}$$

In the second figure, the top arrow is an identity morphism by part (b) and the bottom arrow is an identity morphism since  $m$  belongs to  $\mathcal{M}$ . Thus commutativity of the square implies part (c).

The remaining assertions follow by symmetrical reasonings.  $\square$

For any morphisms  $m : y \rightarrow x$  and  $n : x \rightarrow z$  in  $\mathcal{M}$ , we need to show that

$$\left( F(1_y) \xrightarrow{F(1_y, nm)} F(nm) \right) = \left( F(1_y) \xrightarrow{F(1_y, m)} F(m) \xrightarrow{F(1_y, n)} F(nm) \right)$$

is an identity morphism. This holds because the first arrow on the right hand side is an identity morphism since  $m$  belongs to  $\mathcal{M}$  and the second arrow is an identity morphism by part (b) of Lemma. This proves that  $\mathcal{M}$  is closed under composition and symmetrically, so is  $\mathcal{E}$ .

We have to construct a section for the 2-cell  $\mathcal{EM} \hookrightarrow \mathcal{CC} \xrightarrow{\circ} \mathcal{C}$  in **SetMat**. This is done by putting, for any morphism  $f : a \rightarrow x$  in  $\mathcal{C}$ ,

$$\epsilon(f) := \left( a \xrightarrow{\vartheta_a^{-1}} F(1_a) \xrightarrow{F(1_a, f)} F(f) \right); \quad \mu(f) := \left( F(f) \xrightarrow{F(f, 1_x)} F(1_x) \xrightarrow{\vartheta_x} x \right).$$

In view of (3.6), it gives a factorization  $f = i\mu(f)i\epsilon(f)$  of  $f$  (where  $i$  stands for the obvious inclusions  $\mathcal{E} \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{M}$ ). It follows by the coherence of  $\vartheta$  and associativity of  $F$  that  $F(i\epsilon(f), 1_{F(f)})$  is equal to

$$F \left( i\epsilon(f) \downarrow \begin{array}{ccc} a & \xrightarrow{\vartheta_a^{-1}} & F(1_a) \xrightarrow{F(1_a, f)} F(f) \\ F(f) & \xrightarrow{\vartheta_{F(f)}^{-1} = 1_{F(f)}} & F(f) \end{array} \downarrow \begin{array}{ccc} F(f) & \xrightarrow{F(f, 1_x)} & F(1_x) \xrightarrow{\vartheta_x} x \end{array} \right) = F \left( f \downarrow \begin{array}{ccc} a & \xrightarrow{1_a} & a \\ x & \xrightarrow{1_x} & x \end{array} \right),$$

i.e. to  $1_{F(f)}$ ; and in particular  $F(i\epsilon(f)) = F(f) = F(1_{F(f)})$ . Hence  $\epsilon(f)$  belongs to  $\mathcal{E}$  and symmetrically,  $\mu(f)$  belongs to  $\mathcal{M}$ . It remains to check properties (2.1) and (2.2). Consider any morphisms  $e : x \rightarrow a$  in  $\mathcal{E}$ ,  $f : a \rightarrow b$  in  $\mathcal{C}$  and  $m : b \rightarrow z$  in  $\mathcal{M}$ . Then by part (a') in Lemma,  $F(fe) = F(f)$  and by part (b'),

$$\mu(fi(e)) = \left( F(fe) \xrightarrow{F(e, 1_b) = 1_{F(f)}} F(f) \xrightarrow{F(f, 1_b)} F(1_b) \xrightarrow{\vartheta_b} b \right) = \mu(f).$$

Moreover, by part (a),  $F(mf) = F(f)$  and by part (c),  $\mu(i(m)f)$  is equal to

$$\left( F(mf) \xrightarrow{F(f, 1_z)} F(m) \xrightarrow{F(1_C(m) = F(m, 1_z))} F(1_z) \xrightarrow{\vartheta_z} z \right) = \left( F(f) \xrightarrow{F(f, 1_b)} F(1_b) \xrightarrow{\vartheta_b} b \xrightarrow{m} z \right),$$

i.e. to  $m\mu(f)$ . This proves that  $\mu$  obeys (2.2) and symmetrically,  $\epsilon$  is shown to obey (2.1).

Our next aim is to relate the constructions in 3.3 and 3.4.

**3.5. Strict morphisms of (strictly associative) pseudoalgebras of the 2-monad  $(-)^2$ .** Recall from [6] that a strict morphism  $(\mathcal{C}, F, \vartheta, \varphi) \rightarrow (\mathcal{C}', F', \vartheta', \varphi')$  of pseudoalgebras for the 2-monad  $(-)^2$  is a functor  $Q : \mathcal{C} \rightarrow \mathcal{C}'$  such that the following diagrams (of functors and of natural transformations, respectively) commute.

$$\begin{array}{ccc} \mathcal{C}^2 & \xrightarrow{Q^2} & \mathcal{C}'^2 \\ F \downarrow & & \downarrow F' \\ \mathcal{C} & \xrightarrow{Q} & \mathcal{C}' \end{array} \quad \begin{array}{ccc} QFI_C & \xrightarrow{Q\vartheta} & Q \\ \parallel & & \parallel \\ F'Q^2I_C & & \\ \parallel & & \parallel \\ F'I_{C'}Q & \xrightarrow{\vartheta'Q} & Q \end{array} \quad \begin{array}{ccc} QFM_C & \xrightarrow{Q\varphi} & QFF^2 \\ \parallel & & \parallel \\ F'Q^2M_C & & F'Q^2F^2 \\ \parallel & & \parallel \\ F'M_{C'}Q^{2 \times 2} & \xrightarrow{\varphi'Q^{2 \times 2}} & F'F'^2Q^{2 \times 2} \end{array}$$

If the pseudoalgebras  $(\mathcal{C}, F, \vartheta, \varphi)$  and  $(\mathcal{C}', F', \vartheta', \varphi')$  are strictly associative then the last diagram becomes trivial.

(Strictly associative) pseudoalgebras of  $(-)^2$  and their strict morphisms constitute a category.

**Theorem 3.6.** *There is an adjunction with a trivial counit, between the category of bilinear factorization systems and the category of strictly associative pseudoalgebras of the 2-monad  $(-)^2$  on  $\mathbf{Cat}$ .*

*Proof.* It is easy to see that both constructions in 3.3 and 3.4 can be extended to functors  $L$  and  $R$  between the stated categories, both acting on the morphisms as identity maps.

Let  $(\mathcal{C}, F, \vartheta, 1)$  be a strictly associative pseudoalgebra of  $(-)^2$  and consider the associated bilinear factorization system  $\mathcal{E} \hookrightarrow \mathcal{C} \leftarrow \mathcal{M}$  in 3.4. For any morphism  $(u, v) : f \rightarrow g$  in  $\mathcal{C}^2$  (cf. first figure in (3.1)),

$$i\epsilon(v)i\mu(f) = ( F(f) \xrightarrow{F(f, 1_x)} F(1_x) \xrightarrow{F(1_x, v)} F(v) ) = ( F(f) \xrightarrow{F(f, v)} F(v) ).$$

Hence  $i\epsilon(i\epsilon(v)i\mu(f))$  is equal to

$$\begin{array}{ccc} F(f) & \xrightarrow{F(1_a, 1_x)} & F(f) \\ F(1_a, 1_x) \downarrow & & \downarrow F(f, v) \\ F(f) & \xrightarrow{F(f, v)} & F(v) \end{array} = \begin{array}{ccc} a & \xrightarrow{1_a} & a \\ f \downarrow & & \downarrow vf \\ x & \xrightarrow{v} & y \end{array}$$

by the associativity of  $F$ . Symmetrically,  $i\mu(i\epsilon(g)i\mu(u)) = F(u, 1_y) : F(gu) \rightarrow F(g)$ , hence

$$( F(f) \xrightarrow{i\epsilon(i\epsilon(v)i\mu(f))} F(vf) = F(gu) \xrightarrow{i\mu(i\epsilon(g)i\mu(u))} F(g) ) = ( F(f) \xrightarrow{F(u, v)} F(g) ).$$

This proves that the composite  $LR$  is the identity functor.

Consider now a bilinear factorization system  $\mathcal{E} \hookrightarrow \mathcal{C} \leftarrow \mathcal{M}$  with structure functions  $\epsilon : \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{E})$  and  $\mu : \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{M})$ ; and let  $(\mathcal{C}, F, \vartheta, 1)$  be the associated pseudoalgebra in 3.3. For any morphism  $f : a \rightarrow x$  in  $\mathcal{C}$ ,

$$F(1_a, f)\vartheta_a^{-1} = i\mu(i\epsilon(f)i\mu(1_a)) i\epsilon(i\epsilon(f)i\mu(1_a)) i\epsilon(1_a) = i\epsilon(f)i\mu(1_a)i\epsilon(1_a) = i\epsilon(f).$$

Symmetrically,  $\vartheta_x F(f, 1_x) = i\mu(f)$ , hence  $RL$  takes the bilinear factorization system  $\mathcal{E} \hookrightarrow \mathcal{C} \leftarrow \mathcal{M}$  to

$$\{i\mu(1_x)i\epsilon(f)|a \xrightarrow{f} x; F(f) = F(1_x)\} \hookrightarrow \mathcal{C} \leftarrow \{i\mu(f)i\epsilon(1_a)|a \xrightarrow{f} x; F(f) = F(1_a)\}.$$

For a morphism  $e : a \rightarrow x$  in  $\mathcal{E}$ ,  $i\mu(1_x)i\epsilon(e) = i\mu i(e)i\epsilon(e) = i(e)$ , and for a morphism  $m : a \rightarrow x$  in  $\mathcal{M}$ ,  $i\mu i(m)i\epsilon(1_a) = i\mu i(m)i\epsilon(1_a) = i(m)$ . Thus a natural transformation  $\eta : 1 \rightarrow RL$  is given, at a bilinear factorization system  $\mathcal{E} \hookrightarrow \mathcal{C} \leftarrow \mathcal{M}$ , by the identity functor  $1_{\mathcal{C}}$ . Evaluated at this object,  $L\eta$  is the identity morphism of the pseudoalgebra  $LRL(\mathcal{E} \hookrightarrow \mathcal{C} \leftarrow \mathcal{M}) = L(\mathcal{E} \hookrightarrow \mathcal{C} \leftarrow \mathcal{M})$ . Since the bilinear factorization systems  $R(\mathcal{C}, F)$  and  $RLR(\mathcal{C}, F)$  are equal, for any strictly associative pseudoalgebra  $(\mathcal{C}, F)$ , also  $\eta R$  is an identity natural transformation. This proves that there is an adjunction  $L \dashv R$ , with trivial counit and unit  $\eta$ .  $\square$

**3.7. Orthogonal factorization systems vs pseudoalgebras of  $(-)^2$ .** Orthogonal factorization systems and pseudoalgebras of  $(-)^2$  were shown in [8] to be equivalent notions. Indeed, for an orthogonal factorization system  $\bar{\mathcal{E}} \hookrightarrow \mathcal{C} \hookleftarrow \bar{\mathcal{M}}$ , a pseudoalgebra structure  $F : \mathcal{C}^2 \rightarrow \mathcal{C}$  is constructed using the diagonal fill-in property. A morphism in  $\mathcal{C}^2$  as in the first diagram of (3.1) is taken by  $F$  to the unique diagonal arrow rendering commutative the following diagram,

$$\begin{array}{ccc} & \xrightarrow{egu} & \\ e_f \downarrow & \nearrow & \downarrow m_g \\ & \xrightarrow{vm_f} & \end{array}$$

where  $f = m_f e_f$  and  $g = m_g f_g$  are factorizations in  $\bar{\mathcal{E}}$  and  $\bar{\mathcal{M}}$ . Conversely, for a pseudoalgebra  $(\mathcal{C}, F)$  of  $(-)^2$ , with coherence natural isomorphism  $\vartheta : FI_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$ , there is an orthogonal factorization system

$$\bar{\mathcal{E}} := \{f \in \mathcal{C} \mid F(f, 1_x) \text{ is iso}\} \hookrightarrow \mathcal{C} \hookleftarrow \{f \in \mathcal{C} \mid F(1_a, f) \text{ is iso}\} =: \bar{\mathcal{M}},$$

with factorization  $f = \vartheta_x F(f, 1_x) F(1_a, f) \vartheta_a^{-1}$  of any morphism  $f : a \rightarrow x$  in  $\mathcal{C}$ .

**Corollary 3.8.** *For any bilinear factorization system  $\mathcal{E} \xrightarrow{i} \mathcal{C} \xleftarrow{i} \mathcal{M}$ , there is a canonically associated orthogonal factorization system  $\bar{\mathcal{E}} \hookrightarrow \mathcal{C} \hookleftarrow \bar{\mathcal{M}}$  such that the functors  $i$  factorize through  $\bar{\mathcal{E}}$  and  $\bar{\mathcal{M}}$ , respectively.*

*Proof.* Applying the functor  $L$  in Theorem 3.6 to any bilinear factorization system  $\mathcal{E} \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{M}$ , we obtain a pseudoalgebra structure for  $(-)^2$  on  $\mathcal{C}$ , with the action functor  $F : \mathcal{C}^2 \rightarrow \mathcal{C}$  described in 3.3. By 3.7, there is a corresponding orthogonal factorization system

$$\bar{\mathcal{E}} := \{f \in \mathcal{C} \mid i\mu(f) \text{ is iso}\} \hookrightarrow \mathcal{C} \hookleftarrow \{f \in \mathcal{C} \mid i\epsilon(f) \text{ is iso}\} =: \bar{\mathcal{M}},$$

with factorization  $f = i\mu(f)i\epsilon(f)$  of any morphism  $f$  in  $\mathcal{C}$ . The category  $\mathcal{E} \cong i(\mathcal{E})$  is contained in  $\bar{\mathcal{E}}$  since for any morphism  $e : a \rightarrow b$  in  $\mathcal{E}$ ,  $i\mu i(e) = i\mu(1_b)$  (cf. (2.2)) is an isomorphism with the inverse  $i\epsilon(1_b)$ . Symmetrically,  $\bar{\mathcal{M}} \supseteq i(\mathcal{M}) \cong \mathcal{M}$ .  $\square$

**Example 3.9.** The orthogonal factorization systems associated to the bilinear factorization systems in Examples 2.2 and 2.3 are trivial; that is, in both cases  $\bar{\mathcal{E}} = \mathcal{C}$  and the morphisms in  $\bar{\mathcal{M}}$  are precisely the isomorphisms in  $\mathcal{C}$ . (These orthogonal factorization systems can be obtained also from strict ones in which  $\mathcal{E} = \mathcal{C}$  and  $\mathcal{M}$  contains precisely the identity morphisms in  $\mathcal{C}$ .)

However, the orthogonal factorization system associated to the bilinear factorization system in Example 2.4 is non-trivial: Unless  $q$  (equivalently,  $p$ ) is an isomorphism,  $p$  does not belong to  $\bar{\mathcal{E}}$  (whose non-identity morphisms are  $f, q, fq$  and  $qp = f^{-1}$ ); and  $q$  does not belong to  $\bar{\mathcal{M}}$  (whose non-identity morphisms are  $f, p, pf$  and  $qp = f^{-1}$ ). Thus in general none of the subcategories  $\bar{\mathcal{E}}$  and  $\bar{\mathcal{M}}$  is equal to  $\mathcal{C}$ .

**Corollary 3.10.** *For any weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$  in  $\mathbf{SetMat}$ , there is an orthogonal factorization system  $\bar{\mathcal{E}} \hookrightarrow \mathcal{M}_\varrho \mathcal{E} \hookleftarrow \bar{\mathcal{M}}$  on the associated product category together with identity-on-objects functors  $\mathcal{E} \rightarrow \bar{\mathcal{E}}$  and  $\mathcal{M} \rightarrow \bar{\mathcal{M}}$ .*

*Proof.* Applying the left adjoint functor in Corollary 2.13 to any weak distributive law  $\varrho : \mathcal{E}\mathcal{M} \rightarrow \mathcal{M}\mathcal{E}$ , we obtain a bilinear factorization system  $i(\mathcal{E}) \hookrightarrow i(\mathcal{M})_{S(\varrho)} i(\mathcal{E}) =$

$\mathcal{M}_\rho \mathcal{E} \hookleftarrow i(\mathcal{M})$ , where  $S$  is the left adjoint functor from Proposition 1.10 and  $i$  stands for the identity-on-objects functors in Proposition 1.6. Hence the claim follows by Corollary 3.8.  $\square$

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